

E L E M E N T S
O F
M A T H E M A T I C S.

C O M P R E H E N D I N G

G E O M E T R Y. || M E N S U R A T I O N.
C O N I C S E C T I O N S. || S P H E R I C S.

I L L U S T R A T E D W I T H 30 C O P P E R - P L A T E S.

F O R T H E U S E O F S C H O O L S.

B Y

J O H N W E S T,
A S S I S T A N T T E A C H E R O F M A T H E M A T I C S I N T H E
U N I V E R S I T Y O F S^t. A N D R E W S.

E D I N B U R G H:

P R I N T E D F O R W I L L I A M C R E E C H;

A N D S O L D I N L O N D O N B Y

T. L O N G M A N A N D T. C A D E L L.

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Gift of

Mrs. Eliza Farrar

of the University of Cambridge.

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I am perfectly disposed to gratify this reason-
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P R E F A C E.

SOME account of the following work, and of the motives which induced me to publish it, will perhaps be expected by the reader. I am perfectly disposed to gratify this reasonable expectation; and the more so, as an attempt, which proposes great innovations in the long established system, requires an apology.

My original intention was not to include the First Elements of Geometry. It then appeared to me, that the Elements of Euclid might continue to serve the purpose which they had done for many ages. My design was only to build upon the foundation which that illustrious author had laid, and, under the several heads of *Conic Sections*, *Mensuration*, and *Spherics*, to complete a system of Geometry for the instruction

tion of youth. The expediency and utility of such a work would, I thought, have occurred to every teacher. I even began to wonder that it had not been attempted. Though there are many excellent performances in our own language, on every branch of mathematical science, yet the high price which books of science commonly bear, renders the purchase of them too expensive for general use.

In the execution of this work, I had many difficulties to encounter. As I proposed to establish it entirely upon geometrical principles, I could derive no assistance from some of the best authors on the subjects I treat, who have introduced both Algebra and Fluxions into their demonstrations. Besides, the concise plan which I meant to follow, rendered the works of others, where diffuse, of little advantage to me. I had, therefore, many new demonstrations to investigate; But the greatest difficulty arose from the narrowness of the basis on which I had to build. In the present advanced state of Geometry, there are many useful Theorems, which, though not in Euclid, are considered by most Geometers as elementary propositions.

These

P R E F A C E.

These I could not admit without proof; and, to demonstrate them, I must either have incumbered the demonstrations of almost every proposition, or have interspersed the work with Lemmas. Both of these methods appeared to me equally unsystematical and inelegant. To begin, therefore, from the very foundation, seemed the best method to communicate consistency and order to the system I intended to form.

The Elements of Euclid have continued for two thousand years to hold possession of the Schools. This singular fact, and the bad success of the attempts that have been made to supplant them, have been reckoned convincing proofs of their superior excellence. But these attempts may also serve to support what I suggested above, that, considering the improved state of mathematics, Euclid's Geometry is now inadequate and defective, as an elementary work. The bad success of new elements may be owing partly to the prejudice which strongly prevails in favour of Euclid, and which serves to counterbalance the merit of improvements; but chiefly to the peculiar objections to which their imper-

imperfections seem to be liable. If, besides, it be considered, that the most exceptionable part of the ancient Elements, is that in which succeeding writers have most failed, it will not appear surprising that Euclid's name should have triumphed over partial improvements.

The doctrine of Proportion is perhaps the most important in mathematics, and, as treated by Euclid, discovers great penetration and sagacity. In order to include incommensurables, he assumes, in his definition, a general property of proportionals, very remote from common apprehension. On this property, he builds a very ingenious and comprehensive theory; yet not with all that clearness and solidity for which the rest of his Elements is so remarkable. Indeed, his manner of treating this difficult and abstract subject, is so obscure, as to render it almost unintelligible to the learner. It was a desire to render this intricate and useful doctrine more simple and intelligible, which determined me to attempt a new theory of proportion, and to introduce a new system of the Elements of Geometry as the first part of my work.

What

What I offer to the Public, on the subject of Proportion, will, I trust, sufficiently evince that my labours have not been altogether unsuccessful. And the approbation with which they have been honoured by men of distinguished abilities in mathematical researches, gives me ground to hope that they may be useful. The principle I have assumed is the most simple and obvious; and the theory I have founded upon it, is not less general than that of Euclid.

As to the plan I have followed in the rest of the Elements, I have constantly kept in view the celebrated Ancient. Indeed, I have never departed from him, unless I could give more easy demonstrations, or could substitute more useful or more general theorems. I reflect, with pleasure, that, notwithstanding considerable additions have been made to the former stock of elementary knowledge, yet the number of propositions is so much abridged, as to render the study of it a task of much less labour and difficulty.

I am aware of an objection, arising from the conciseness of this work. It will be said, perhaps, that many propositions are left undemonstrated, and annexed as corollaries to others, while

while their dependence on these is not obvious to a beginner. But I have to observe, that such propositions serve to sharpen the genius, and to exercise the invention, of youth; and that, considering how much they are accustomed to exercises of memory in the first branches of education, it is of the utmost importance to engage them now to exert the powers of the understanding. This exertion cannot be effected by long formal demonstrations. Many students estimate the difficulty of their task by its length; they wish to continue to lay the burden on their memory; and they imagine, that to repeat is the same thing as to comprehend. That they may be induced, therefore, to apply all the powers of reason, and may reap the most advantage from such application, I have thought it requisite, not only to make the style of demonstration simple and concise, but also to leave such propositions, as easily follow from those that are demonstrated, without further evidence of their truth: For, if the evidence arising from thence be explained, when necessary, by the teacher, they will be much sooner apprehended than in the method of formal demonstration. To illustrate this remark, suppose the learner perfectly

perfectly master of the Fourth Proposition of the First Book of Euclid, I affirm that he will be more easily convinced of the truth of the following proposition, by deducing it from thence, and imagining the triangles to be isosceles, than by the long and intricate demonstration of Euclid. I would not, however, be understood to insinuate that equal advantage will be gained in every instance in which I have converted into corollaries the propositions which Euclid has demonstrated. I only presume to infer, from this and other instances which might be adduced, that the method here pursued, instead of being incumbered with difficulties, as Euclid's admirers will be apt to alledge, will tend much to facilitate rational and useful progress in the Elements.

In the Sixth Book, which treats of solids, I have omitted the propositions respecting the measuring of the sphere, first, because they are demonstrated, in the Third Book of Mensuration, in a manner, not consistent perhaps with the ancient strictness of Geometry, but equally convincing; and with much more conciseness, than a regard to that strictness, which I have

always endeavoured to preserve in the Elements, would have admitted; and, secondly, because I would refer the young proficient in Geometry to Archimedes's Treatise on the Sphere and Cylinder, where, in the simple and accurate, though prolix method, of ancient demonstration, his mind will trace with peculiar pleasure the very steps by which that illustrious philosopher arrived at these important discoveries.

Throughout the whole work, I have endeavoured to adopt the arrangement which is most systematic, not that which is commonly observed in teaching; because the latter may be varied, while the former, remaining the same, tends to give clear and distinct views of the several subjects. I have also interspersed a great variety of Problems and Theorems, which, tho' intended for exercises to the learner, are not unworthy the notice of those who are farther advanced, and take pleasure in mathematical speculations.

E L E M E N T S

O F

G E O M E T R Y.

B O O K I.

D E F I N I T I O N S.

1. **G**EOMETRY is the science in which the properties and relations of those quantities which have extension, called Magnitudes, are investigated.

Extension is distinguished into length, breadth, and thickness.

2. A line is that which has only length. The bounds or extremes of a line are points.

3. A surface is that which has only length and breadth.

The bounds of a surface are lines.

A

4. A

4. A right or straight line is that which every where tends the same way.

5. A plane surface is that in which any two points being taken, the straight line that joins them lies wholly in that surface.

6. The opening between two straight lines which meet in the same point, is called an angle.

7. When one straight line, standing upon another, makes the adjacent angles equal, each of them is called a right angle, and the former line is said to be perpendicular to the latter.

8. An acute angle is that which is less than a right angle.

9. An obtuse angle is that which is greater than a right angle.

10. The distance of two points is the straight line joining them. And the distance of a point from a straight line is a perpendicular drawn to the line from that point.

11. Parallel straight lines are such as lie in the same plane, and, though produced to any length both ways, do not meet.

12. A figure is a bounded space. When it is a surface, it is called a plane figure.

13. A plane rectilineal figure is that which is bounded by straight lines.

14. Every plane figure, bounded by three straight lines, is called a triangle, of which the three straight lines

lines are the sides. That side upon which the triangle is conceived to stand, is named the base, and the opposite angular point the vertex.

15. An isosceles triangle is that which has two sides equal.

16. An equilateral triangle is that which has all its sides equal.

17. A right angled triangle is that which has a right angle. The side opposite to the right angle is called the hypotenuse.

18. An obtuse angled triangle is that which has an obtuse angle.

19. Every plane figure, bounded by four straight lines, is called a quadrangle or quadrilateral, of which the straight line joining two opposite angles is called a diagonal.

20. A parallelogram is a quadrilateral, of which the opposite sides are parallel.

21. A rectangle is a parallelogram which has all its angles right.

22. A rhombus is a parallelogram which has all its sides equal.

23. A square is a parallelogram which has all its sides equal, and all its angles right.

24. Every plane figure, bounded by more than four straight lines, is called a polygon.

25. A circle is a plane figure bounded by one line, called the circumference, which is such, that all straight

straight lines drawn to it, from a certain point within the figure, called the center, are equal ; and these straight lines are named the radii of the circle.

P O S T U L A T E S.

1. Let it be granted, that, from any given point, to any other given point, a straight line may be drawn.
2. That a terminated straight line may be produced to any length.
3. That a circle may be described about any center at any distance from that center.
4. And that any magnitude, however small, may be multiplied so as to exceed a given magnitude of the same kind.

A X I O M S.

1. Those magnitudes which, if applied to one another, would coincide, or exactly fill the same space, are equal.
2. Things that are equal to the same are equal to one another.
3. If equals be added to equals, the wholes are equal ; and, if equals be taken from equals, the remainders are equal.

4. If

4. If equals be added to unequals, or taken from them, the sums, or remainders, will still be unequal, and have the same difference.

5. The doubles, or halves of equal things, are equal.

6. Every whole is greater than its part, and equal to all its parts taken together.

7. From any one point, to any other point, more than one straight line cannot be drawn.

8. All right angles are equal to one another.

9. A straight line cannot first come nearer to a straight line, and then go farther from it, without cutting it; nor can a straight line first recede from another, and then come nearer to it.

PROPOSITION

PROPOSITION I. PROBLEM.

Upon a given straight line to describe an equilateral triangle.

Fig. 1.

LET AB be the given straight line, upon which it is required to describe an equilateral triangle.

^a post. 3.

About the centers A, B, with the same radius AB, let the circles BCD and ACE be described^a; and, from the point C, where they intersect, let the straight

^b post. 1.

lines CA, CB, be drawn to the points A, B^b; then ABC is an equilateral triangle.

^c def. 25.

For AC is equal to AB^c, because they are radii of the same circle; BC is equal to AB for the same

^d ax. 2.

reason; and, by consequence, AC is equal to BC^d.

PROP. II. PROB.

From a given point to draw a straight line equal to a given straight line.

Fig. 2.

Let A be the given point, and BC the given straight line; it is required to draw from A a straight line equal to BC. Let the straight line AB

^a post. 1.

be drawn^a, and the equilateral triangle ADB described

ed upon it^b. About the center B, with the radius ^b prop. 1.
 BC, let the circle CEF be described ^c, intersecting DB ^c post. 3.
 produced in E^d. And about the center D, with the ^d post. 2.
 radius DE, let the circle EGH be described, inter-
 secting DA produced in G. Then AG is a straight
 line drawn from A equal to BC. For, the straight
 lines DG, DE are equal, because they are radii of the ^e def. 25.
 same circle, and DA, DB parts of these are equal ^e,
 because they are sides of an equilateral triangle; there-
 fore the remainder AG is equal to the remainder BE^f. ^f ax. 4.
 But BC is likewise equal to BE; for BC, BE are
 radii of the circle CEF; consequently AG is equal
 to BC ^g. ^g ax. 2.

PROP. III. PROB.

*From the greater of two given straight lines to cut off a
 part equal to the less.*

Let AB and C be the two given straight lines, of Fig. 3.
 which AB is the greater; it is required to cut off
 from AB a part equal to C. From the point A let
 there be drawn the straight line AD equal to C^a, ^a prop. 2.
 and about the center A, with the radius AD, let the
 circle DEF be described, intersecting AB in E:
 Then AE is a part cut off from AB, equal to C. For
 For

For AE is equal to AD, because they are radii of the same circle ; and C is equal to AD by construction ; therefore AE is equal to C^b.

^b ax. 2.

P R O P. IV. T H E O R E M.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by these sides equal, they shall be equal in every respect.

Fig. 4.

Let ABC, DEF be two triangles, having the side AB equal to DE, AC equal to DF, and the angle BAC equal to DEF ; then shall the remaining sides BC, EF be equal, and the angles subtended by equal sides equal, viz. ABC to DEF, and ACB to DFE ; and the whole triangle ABC shall be equal to the whole triangle DEF.

For, since the angles BAC, EDF are equal, if they were applied to one another they would coincide ; the side AB would fall upon DE, and AC upon DF^a. At the same time, the point B would fall upon E, because AB is equal to DE ; and the point C upon F, because AC is equal to DF^a. Consequently the straight line BC would coincide with EF^b, the angle ABC with DEF, the angle ACB with DFE, and the whole triangle ABC with the whole triangle DEF.

^a ax. 2.

^b ax. 7.

DEF.

DEF. And therefore the triangles must be equal in every respect *. ax. 2.

COROLLARY. If two sides of a triangle be equal, the angles opposite to them are equal.

PROP. V. THEOR.

If two triangles have two angles of the one equal to two angles of the other, each to each, and have likewise the sides between these angles equal, they shall be equal in every respect.

Let ABC, DEF be two triangles, having the angle ABC equal to DEF, the angle ACB equal to DFE, and the side BC equal to EF; then shall the remaining angle BAC be equal to the remaining angle EDF, and the sides subtending equal angles equal; viz. AB to DE, and AC to DF, and the whole triangle ABC shall be equal to the whole triangle DEF. Fig 4.

For, since the sides BC, EF are equal, if they were applied to one another they would coincide *. ax. 1. And, at the same time, the triangle ABC being applied to DEF, the side AB would fall upon DE, because the angle ABC is equal to DEF, and the side AC upon DF, because the angle ACB is equal to DFE. Therefore the point A, where AB and AC meet,

def. 4. meet, would fall upon D^b. And thus the side AB would coincide with DE, the side AC with DF, the angle BAC with EDF, and the whole triangle ABC with the whole triangle DEF. Consequently these triangles are equal in every respect.

COR. If two angles of a triangle be equal, the sides opposite to them are equal.

PROP. VI. THEOR.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, they shall be equal in every respect.

Fig. 5.
6.
7.

Let ABC, DEF be two triangles, having the side AB equal to DE, AC to DF, and BC to EF; then the angles subtended by equal sides shall be equal; viz. ABC to DEF, ACB to DFE, and BAC to EDF, and the whole triangle ABC shall be equal to the whole triangle DEF.

For, since the bases BC, EF are equal, if applied to one another they would coincide^a; and then, the triangles being on different sides of their common base, the straight line DA, which joins their vertices, would intersect the common base EF, or that line produced, or would pass through one of its extremities.

If

If DA intersected EF, or EF produced, it would form with the two equal sides AB, DE a triangle DBA, having the angle BAD equal to BDA^b; and with the two equal sides AC, DF a triangle DCA, having the angle CAD equal to CDA^b. Therefore the whole or remainder, the angle BAC, is equal to the whole or remainder, the angle EDF^c.

If DA passed through the point F, the sides AC, DF would be in one straight line, and the angle BAD equal to BDA^b, that is, BAC to EDF, as before.

Consequently these triangles are equal in every respect^d.

PROP. VII. PROB.

To bisect a given angle.

Let ABC be the given angle; it is required to divide it into two equal parts.

In AB let any point D be assumed, and from BC let BE be cut off equal to BD^a; let DE be drawn, the equilateral triangle DFE be described upon DE^b, and the points B, F joined. The straight line BF bisects the angle ABC.

For, let the triangles BDF, BEF be compared; the side BD is equal to BE by construction, DF, FE are equal,

equal, because they are sides of an equilateral triangle, and BF is common to both; therefore
 * pr. 6. these triangles are equal in every respect. Thus, the angles DBF, FBE subtended by DF, FE are equal, and the whole angle ABC is bisected by the line BF.

PROP. VIII. PROB.

To bisect a given straight line.

Fig. 9. Let AB be the given straight line; it is required to divide it into two equal parts.

Upon AB let the equilateral triangle ABC be described^a, and let the angle ACB be bisected by the
 * pr. 1.
 b. pr. 7. straight line CD^b; CD also bisects AB.

For the triangles ACD, BCD having AC equal to CB, CD common to both, and the angle ACD equal
 c. pr. 4. to BCD, are equal in every respect. Thus, AD is equal to DB, and the whole line AB bisected in D.

COR. The straight line which bisects the vertical angle of an isosceles triangle, bisects also the base at right angles.

PROP.

PROP. IX. PROB.

To draw a straight line perpendicular to a given straight line from a given point in the same.

Let AB be the given straight line, and C the given point in it; it is required to draw from C a straight line perpendicular to AB. Fig. 10.

In AC let any point D be assumed, and from CB let CE be cut off equal to CD^a. Let the equilateral triangle DFE be described upon DE^b, and the points F, C be joined. The straight line FC is perpendicular to AB. ^a pr. 3.
^b pr. 1.

For the triangles FDC, FEC having FD equal to FE, DC equal to CE, and FC common to both, are equal in every respect^c. Thus, the angle DCF is equal to ECF, that is, the angles at C are right^d, and FC is perpendicular to DE or AB^d. ^c pr. 6.
^d def. 7.

COR. A line drawn from the vertex of an isosceles triangle to bisect the base, is perpendicular to it, and bisects the vertical angle.

PROP.

PROP. X. PROB.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Fig. 11. Let AB be the given straight line, and C the given point without it; it is required to draw from C a straight line perpendicular to AB.

Let any point D be assumed on the other side of AB, and about the center C, with the radius CD, let the circle EDF be described, meeting AB in the points E, F. Let EF be bisected in G^a, and the points G, C be joined; then CG is perpendicular to AB.

For, the radii EC, CF being drawn, ECF is an isosceles triangle^b, and CG a line drawn from the vertex to bisect the base; CG is therefore perpendicular to EF, or AB^c.

PROP. XI. THEOR.

The angles, which one straight line makes with another, upon one side of it, are together equal to two right angles.

Let

Let the straight line AB make with CD , upon one side of it, the angles ABC , ABD ; these shall be together equal to two right angles. Fig. 12.

If AB be perpendicular to CD , then ABC , ABD are two right angles. If not, let BE be perpendicular to it^a, and BE divides the greater angle ABC * pr. 9. into the two parts EBC , EBA . Now, the one part EBC being a right angle^b, and the other part EBA , added b def. 7. to the less angle ABD , being equal^c to another right angle^b; it is plain that the whole angle ABC , added c ax. 6. to ABD , is equal to two right angles.

COR. 1. All the simple angles formed upon one side of a straight line, at the same point in it, are together equal to two right angles.

COR. 2. All the simple angles, formed about the same point, are together equal to four right angles.

COR. 3. The converse of the proposition. If two straight lines, drawn from the extremity of another, make with it two angles, which are together equal to two right angles, they are in the same straight line.

If ABC , ABD be together equal to two right Fig. 13. angles, BD is in the same straight line with CB . For, if not, let BF be the continuation of CB , then ABC , ABF would be equal to two right angles, and therefore equal to ABC , ABD ; which is impossible.

P R O P.

Let

PROP. XII. THEOR.

If two straight lines cut one another, the opposite angles are equal.

Fig. 14.

Let the two straight lines AB, CD cut each other in E, the angle AEC shall be equal to BED, and CEB to AED. For the angles which AE makes with CD are together equal to two right angles^a; and the angles which DE makes with AB are also equal to two right angles^a; therefore the two former AEC, AED are together equal to the two latter AED, DEB^b. And the angle AED, which is common, being taken away, there remains the angle AEC equal to DEB. In the same manner, CEB may be proved equal to AED.

^a ax. 2.

^a cor. 3.

11.

COR. If from the same point, in a straight line, two others be drawn on different sides of it, making the opposite angles equal, they shall be in the same straight line^c.

PROP. XIII. PROB.

From a given point, in a given straight line, to draw another straight line, that shall make with the former an angle equal to a given angle.

Let

Let AB be the given line, and CDE the given angle; it is required to draw a line from the point A , that shall make with AB an angle equal to CDE . Fig. 15.

If CDE be a right angle, let AK be drawn perpendicular to AB , and it is done. ^a pr. 9.
 If not, from any point G , in CD , let a perpendicular GF be drawn to DE , produced if necessary ^b. Then, from ^b pr. 10.
 AB let AH be cut off equal to DF ^c, let the perpendicular HK be drawn ^a equal to FG ^c, and let the points A, K be joined. The triangle KAH , formed in this manner, will be equal in every respect to the triangle DGF ^d. Therefore the angle KAB will be equal to CDE , and consequently KAB to CDE . ^d ax. 8.
^e pr. 4.
^e pr. 11.

P R O P. XIV. T H E O R.

If a straight line, falling upon two other straight lines, make the alternate angles equal, these two lines are every where equally distant from one another.

Let the straight line EF , falling upon the two straight lines AB, CD , make the alternate angles AEF, EFD equal; then AB, CD are every where equidistant. Fig. 16.
17.

CASE I. When EF is perpendicular to the lines AB, CD ; the distance AC of any point A in one of them, from the other CD , shall be equal to EF .

C

For,

Let

For, let EB be equal to EA , FD to FC , and B, D joined; then, if the angle EFC were applied to EFD ,

- ^a ax. 1. and FEA to FEB , FC would fall upon FD ^a, because the angles at F are equal, and the point C upon D ^a, because the lines FC, FD are equal. In like manner, the point A would fall upon B . Therefore the straight line AC would coincide with BD ^b, and the angle ACF with BDF . Consequently BD is perpendicular to CD , and equal to AC . And, since the distances AC, BD are equal to one another, it is evident that each of them is equal to EF ^c.

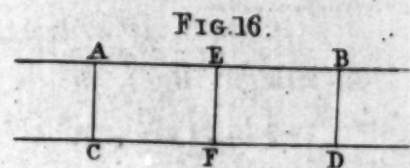
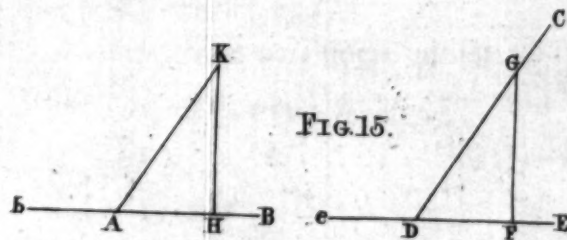
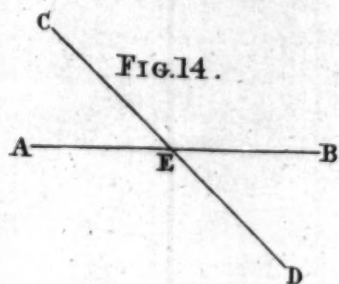
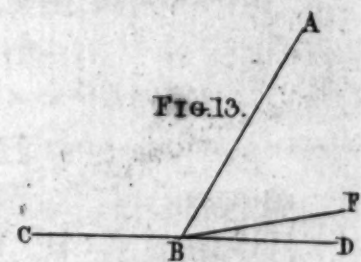
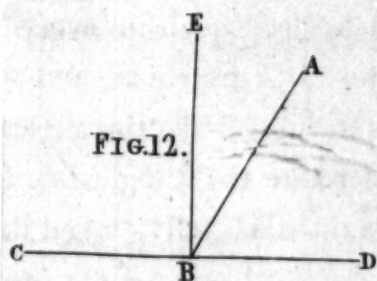
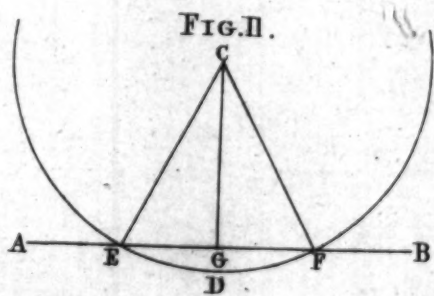
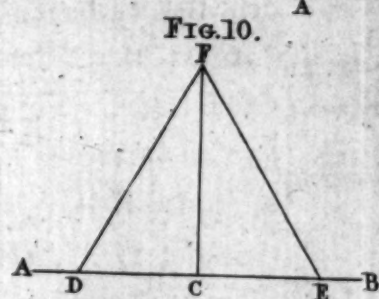
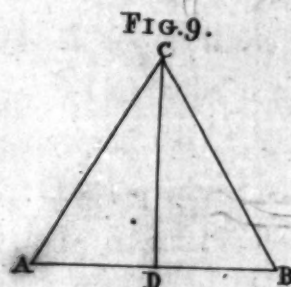
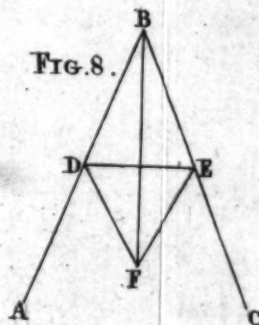
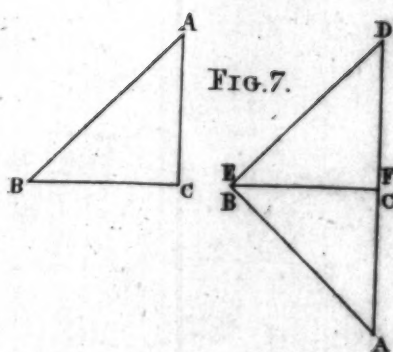
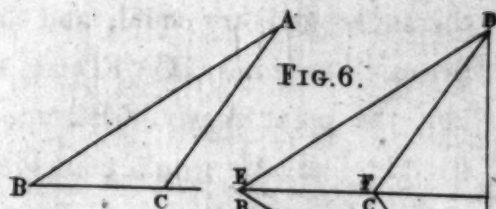
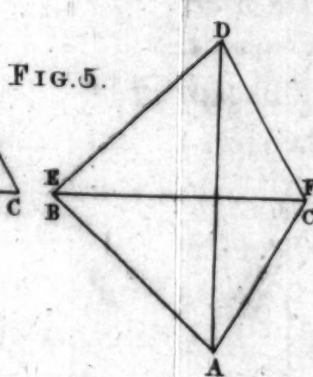
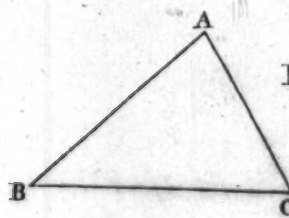
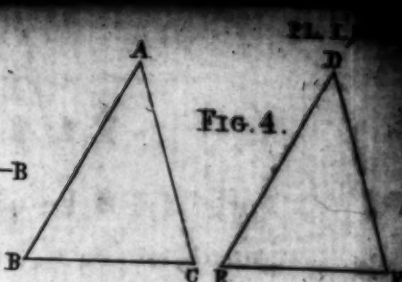
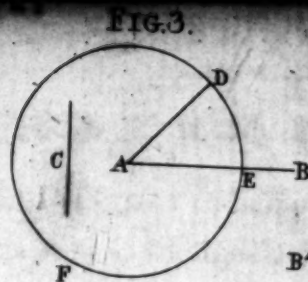
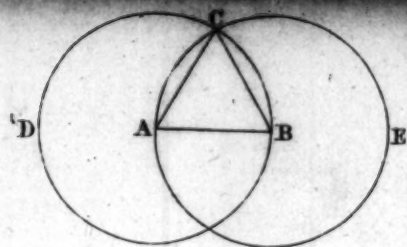
^b ax. 7. Fig. 17. CASE II. When EF is not perpendicular to AB ,

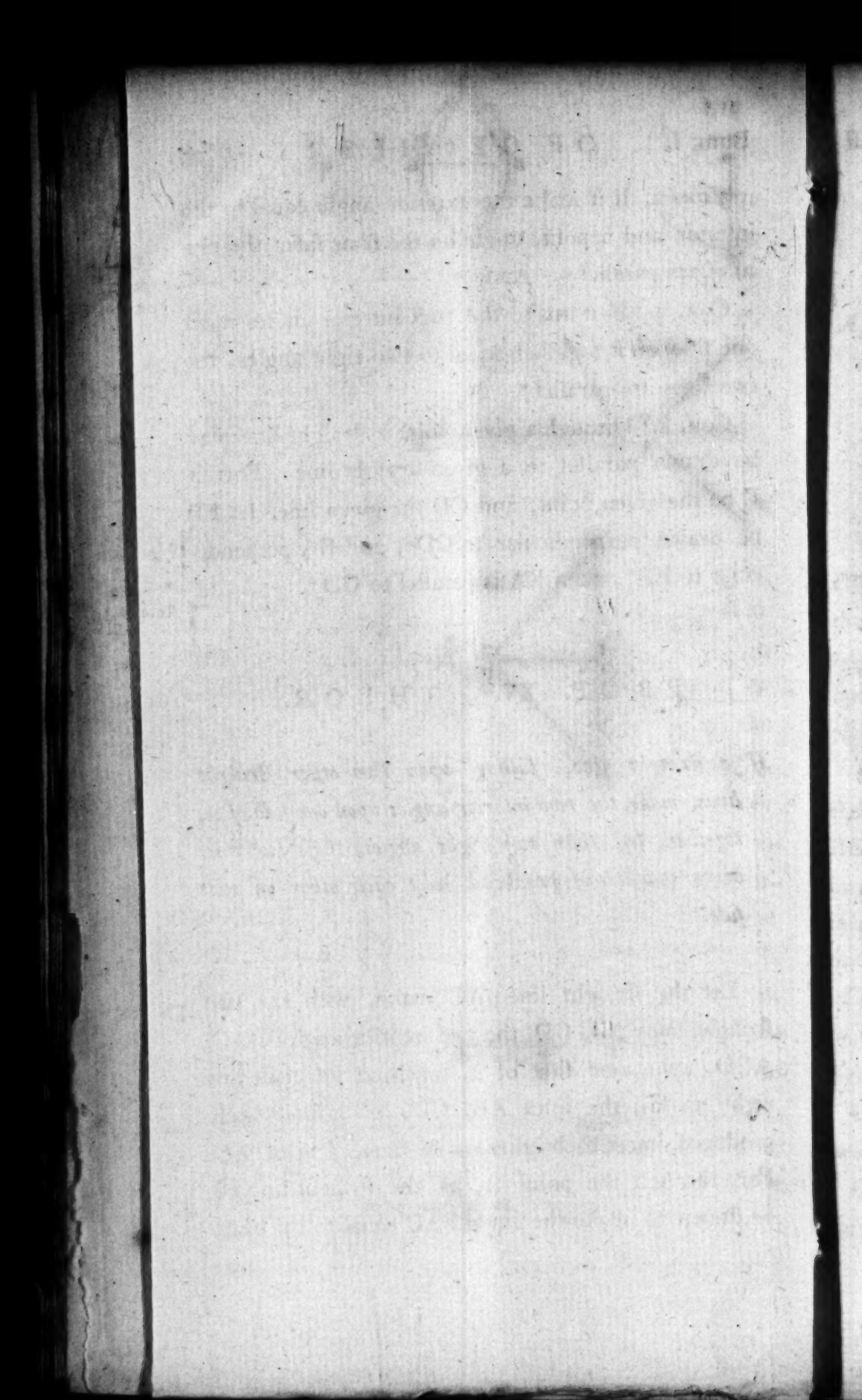
^d pr. 8. CD ; let EF be bisected in the point G ^d, from which

^e pr. 10. let GH be drawn perpendicular to AB ^e, also let FK be equal to HE , and the points G, K joined. The triangles GEH, GFK , having the angles at E, F equal by hypothesis, and the containing sides equal by construction, each to each, are equal in every respect^f. Hence the angles at G are equal, and consequently HGK a straight line^g; also the angles at H and K equal, and therefore the straight line HK perpendicular to both the lines AB, CD . And thus, by the First Case, the distance of any point, in one of the lines AB, CD , from the other, is equal to HK .

^f pr. 4. COR. I. If a straight line, falling upon two other straight lines, make the alternate angles equal, the two lines are parallel^h. Hence,

COR.





COR. 2. If it make the exterior angle equal to the interior and opposite angle on the same side, the two lines are parallel ^l. And,

^l pr. 12.
cor. 1. 14.

COR. 3. If it make the two interior angles upon one side of it together equal to two right angles, the two lines are parallel ^k.

^k pr. 11.
cor. 1. 14.

COR. 4. Through a given point a straight line may be drawn parallel to a given straight line. For, if E be the given point, and CD the given line, let EF be drawn perpendicular to CD ^l, and EA perpendicular to EF ^m, then EA is parallel to CD ⁿ.

^l pr. 10.
^m pr. 9.
ⁿ cor. 1. 14.

P R O P. XV. T H E O R.

If a straight line, falling upon two other straight lines, make the two interior angles upon one side of it, together, less than two right angles, these two lines being indefinitely produced, meet each other on that side.

Let the straight line AC make, with the two straight lines AB, CD, the two interior angles BAC, ACD, upon one side of it, together, less than two right angles, the lines AB, CD, being indefinitely produced, meet each other on the same side of AC. For, through the point A, let the straight line FI be drawn to make the angle FAC equal to the alter-

Fig. 18.

nate

- ^a pr. 11. nate Angle ACD; then the angles LAC, ACD are together equal to two right angles^b, and therefore the angle LAC greater than BAC. Hence AB is on the same side of FL with CD. From any point E in CD, and from B in AB, let the perpendiculars EF,
- ^b pr. 10. BH be drawn to FL^b. In AB and HB produced, let BK be taken equal to BA; and BG equal to BH; also let G, K be joined, and KL be perpendicular to AL. The triangles AHB, GBK have the opposite
- ^c pr. 12. angles at B equal^c, and the containing sides equal by construction, each to each; therefore the angles at
- ^d pr. 4. H and G are equal^d, and consequently KL equal to
- ^e pr. 14. GH^e; that is, the distance of K from AL is the double of BH. In like manner, a point may be found in AK produced, whose distance from AL shall be the double of KL; and so on. Consequently a point may be found in the same line AB such, that
- ^f post. 4. its distance from AL shall be greater than EF^f. Let M be such a point, and let the perpendicular MN, upon AL, meet CD in O. Then, because EF and ON are equal, MN is also greater than ON; therefore the point M is on the other side of CD with respect to the point A, and the line ABM passes through CDO.

P R O P.

PROP. XVI. THEOR.

A straight line falling upon two parallels makes the alternate angles equal, the exterior angle equal to the interior opposite angle upon the same side, and the two interior angles upon the same side together equal to two right angles.

Let the line EF fall upon the two parallels AB, CD, Fig. 19. the alternate angles AGH, GHD shall be equal, the exterior angle EGB shall be equal to the interior opposite angle GHD, and the two interior angles upon the same side BGH, GHD shall be equal to two right angles.

For, if possible, let AGH be greater than GHD, then BGH, GHD are together less than AGH, BGH, that is, less than two right angles; and therefore the lines AB, CD, being produced towards B, D, meet one another^a, which is contrary to the hypothesis. Consequently the angles AGH, GHD must be equal. From which it follows that EGB is equal to GHD^b, and BGH, GHD together equal to two right angles^c.

COR. 1. A straight line perpendicular to one of two parallels, is also perpendicular to the other.

COR. 2. A straight line which meets one of two parallels will also meet the other.

COR.

^a pr. 15.

^b pr. 12.

ax. 2.

^c pr. 11.

ax. 2. 3.

Cor. 3. Straight lines, parallel to the same straight line, are parallel to one another.

PROP. XVII. THEOR.

If two lines which meet, be respectively parallel to two other lines, which also meet, and all of them being considered as drawn from the points of concurrence, if both the former be in the same direction with both the latter, or both in opposite directions, the angles they contain shall be equal.

Fig. 20.
21.

Let the two lines AB, AC, which meet in A, be respectively parallel to the two lines DE, DF, which meet in D, and being considered as drawn from the points A, D, let them be all in the same direction, or both AB, AC in opposite directions to DE, DF; the angle BAC shall be equal to EDF.

* cor. 2. 16.

pr. 16.

For, let DF meet AB in G. Then each of the angles BAC, EDF is equal to the same angle BGF, and therefore they are equal to one another.

PROP.

PROP. XVIII. THEOR.

If one side of a triangle be produced, the exterior angle is equal to the two interior opposite angles; and all the three angles of a triangle are together equal to two right angles.

Let ABC be a triangle, having the side BC produced to D ; the exterior angle ACD is equal to the two interior opposite angles CAB, ABC . Fig. 22.

For, through the point C let CE be drawn parallel to AB ^a. Then the line CE divides the angle ACD into the two parts ACE, ECD , of which the former is equal to CAB , and the latter to ABC ^b. ^a cor. 4.
14.
^b pr. 16.

Therefore the whole ACD is equal to CAB and ABC taken together. And to these equals the common angle ACB being added, the three angles of the triangle are equal to the two angles ACD, ACB , and consequently equal to two right angles^c. ^c pr. 11.

COR. 1. The exterior angle of every triangle is greater than either of the two interior opposite angles—any two angles are together less than two right angles—and two of the angles at least are acute.

COR. 2. If two angles of one triangle be, either separately or together, equal to two angles of another, the remaining angles are equal.

COR.

COR. 3. In a right angled triangle, the two acute angles are together equal to a right angle—and, in an isosceles right-angled triangle, each of them is half a right angle.

COR. 4. Each of the angles of an equilateral triangle is one third of two right angles, or two thirds of one right angle.

COR. 5. The four angles of every quadrilateral are together equal to four right angles.

PROP. XIX. THEOR.

In every triangle the greater side subtends the greater angle.

Fig. 23.

Let ABC be a triangle, having the side AB greater than the side AC; then shall the angle ACB opposite to the former be greater than the angle ABC opposite to the latter. For, let AD be cut off from AB, equal to AC, and let C, D be joined. Then the angle ACD is equal to ADC*, and ADC greater than ABC^b; therefore ACD (which is only a part of ACB,) is greater than ABC. Consequently the whole ACB is greater than ABC.

* cor. 4.

^b cor. 1.
18.

COR. 1. Conversely, the greater angle is subtended by the greater side.

COR.

COR. 2. The hypotenuse is the greatest side of a right angled triangle.

COR. 3. Of all straight lines drawn from the same point, to terminate in the same straight line, the perpendicular is the least, and that which is nearer the perpendicular is greater than that which is more remote. Fig. 24.

PROP. XX. THEOR.

Any two sides of a triangle are together greater than the third side.

Two sides AB, AC of the triangle ABC are together greater than the third side BC. For, in AB produced, let AD be taken equal to AC, and C, D joined. Then the angle BCD is greater than ACD^a, and ACD is equal to ADC^b, therefore BCD is greater than ADC or BDC. Consequently BD (which is the sum of AB, AC,) is greater than BC^c. Fig. 25.

COR. Any side of a triangle is greater than the difference between the other two.

D

PROP.

PROP. XXI. THEOR.

If there be two triangles constituted upon the same base, and the one be included in the other, the two sides of the former shall be together less than the two sides of the latter, but shall contain a greater angle.

Fig. 26.
17.

Let ABC, DBC , be two triangles constituted upon the same base BC , of which DBC is included in ABC , the two sides BD, DC , shall be together less than the two sides BA, AC , but the angle BDC greater than BAC .

Fig. 26.

CASE I. When D , the vertex of the triangle BDC , is in AC one of the sides of the triangle ABC ; because BA, AD , are together greater than BD^a , if DC be added to both, BA, AC , are together greater than BD, DC ; and because BDC is an exterior angle of the triangle BDA , it is greater than the interior opposite angle BAC^b .

^a pr. 20.

^b cor. 1.
18.

Fig. 27.

CASE II. When the point D is within the triangle ABC . Let BD be produced to meet AC in E ; then, by CASE I. BA, AC , are greater than BE, EC , and BE, EC greater than BD, DC ; consequently BA, AC , are greater than BD, DC . In like manner, the angle BDC is greater than BEC , and BEC greater than BAC ; consequently BDC is greater than BAC .

COR.

COR. If BA be equal to BD, then CA is greater than CD.

PROP. XXII. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, that which has the greater angle shall likewise have the greater base.

Let ABC, DEF, be two triangles, having AB equal to DE, and AC to DF, but the angle BAC greater than EDF; then shall the base BC be greater than the base EF. Fig. 28.

For, let the angle BAG be made equal to EDF^a, the side AG equal to DF or AC, and B, G joined; then BG and EF are equal^b. Now, if the triangle BGA be included in the triangle BCA, since AC is equal to AG, BC must be greater than BG^c or EF. If not, let G, C be joined, then the angle AGC is is equal to ACG; therefore the angle BGC, which is greater than the former, will be still greater than BCG, which is only part of the latter. And hence BC, as before, is greater than BG^d, or EF.

COR. If two triangles have two sides of the one equal to two sides of the other, each to each, but their bases unequal, that which has the greater base will also have the greater vertical angle.

PROP.

PROP. XXIII. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles subtended by two equal sides equal, and if the angles subtended by the other equal sides be both acute, or both obtuse, or one of them right; then shall the two triangles be equal in every respect.

Fig. 29. Let ABC, DEF be two triangles, having the sides AB, AC equal to the sides DE, DF , each to each, and the angles at B and E opposite to the equal sides AC, DF , also equal; then, if the angles C, F , opposite to the other equal sides, be both acute, the triangles shall be equal in every respect. For, if the remaining sides BC, EF be not equal, let BC be the greater, and the part BG be cut off equal to EF , and A, G joined. The triangles ABG, DEF having the angles at B and E equal, and the including sides equal, each to each, are equal in every respect^a. Thus, the side AG , being equal to DF , will be equal to AC , and the angle AGB being equal to DFE , will be an acute angle: Hence ACG will be an obtuse angle^b; but ACG is an acute angle; therefore AG is not equal to AC ^c, which is repugnant. The same contradiction will follow from supposing BC, EF unequal, when the angles C, F are both obtuse, or when

^a pr. 4.

^b pr. 11.

^c cor. 1.
12.

when one of them is a right angle. The sides BC, EF must therefore be equal, and consequently the triangles equal in every respect^a.

^a pr. 4.

COR. Two right angled triangles having equal hypotenuses, and also one side equal to one side, are equal in every respect.

P R O P. XXIV. T H E O R.

Straight lines which join the extremities of two equal and parallel lines towards the same parts, are themselves equal and parallel.

Let AB, CD be two equal and parallel straight lines, joined towards the same parts by the lines AC, BD; then shall AC, BD be also equal and parallel. For, let AD be drawn; the two triangles ABD, ACD have the side AB equal to CD, by hypothesis, and AD common to both, and the included angles BAD, ADC are equal, because they are alternate angles^a; therefore these triangles are equal in every respect^b. Thus, AC is equal to BD, and the angle CAD to ABD; whence AC is also parallel to BD^c.

Fig. 30.

^a pr. 16.

^b pr. 4.

^c cor. 1.

14.

A P P E N.

APPENDIX.

PROP. 1. If two straight lines be drawn to bisect the angles at the base of a triangle, the straight line which joins the point, where they intersect, and the vertex, will bisect the vertical angle.

PROP. 2. To describe a triangle, having its sides respectively equal to three given straight lines, any two of which are together greater than the third.

PROP. 3. If the two sides of a triangle be bisected by perpendiculars, the base will also be bisected by a perpendicular falling upon it from the point where the two former intersect.

PROP. 4. To describe a right angled triangle, of which the hypotenuse and one side taken together shall be equal to a given straight line, and the remaining side equal to another given line.

PROP. 5. In any triangle, the angle contained by the perpendicular falling from the vertex upon the base, and the straight line bisecting the vertical angle, is equal to half the difference of the angles at the base.

PROP. 6. In a given straight line, of an unlimited length, to find a point equally distant from two given points, which are so situated, that the straight line passing through them does not cut the given line at right angles.

PROP.

PROP. 7. If, from the vertex of a triangle, two straight lines be drawn to meet the base, so as to form two isosceles triangles, having, for their vertical angles, the angles at the base of the triangle, they shall contain an angle equal to half the sum of the angles at the base.

PROP. 8. To construct a triangle, having the angles at the base equal to two given angles, (which are together less than two right angles), each to each, and the sum of all its sides equal to a given straight line.

PROP. 9. If, from the vertex of a triangle, two straight lines be drawn to meet the base, so as to form two isosceles triangles, having for their vertical angles one of the angles at the base, and the exterior angle adjacent to the other, they shall contain an angle equal to half the vertical angle of the triangle.

PROP. 10. A straight line, of an unlimited length, and two points without it being given, to find the point or points in the given line, at which straight lines from the given points shall make equal angles with it.

PROP. 11. All the interior angles of any rectilinear figure are together equal to twice as many right angles, abating four, as the figure has sides.

PROP. 12. All the exterior angles of any rectilinear figure, formed by producing one side from each of the angular points, are equal to four right angles.

PROP.

PROP. 13. If every two adjacent sides of any polygon be produced to intersect the side or sides opposite to them ; the sum of all the salient angles at the points of intersection shall be equal to four right angles, if the number of sides be even, but to two right angles only, if the number of sides be odd.

PROP. 14. The straight line which joins the middle of the base, and the vertex of a triangle, is greater, equal, or less than half the base, according as the vertical angle is acute, right, or obtuse.

PROP. 15. If the base of a right angled triangle be divided into any number of equal parts, and, from the points of section, straight lines be drawn to the opposite angular point, these will divide the angle contained by the hypotenuse and perpendicular into unequal parts, of which that next the perpendicular is the greatest, and the nearer to it is greater than the more remote.

PROP. 16. If two sides of a triangle contain two thirds of a right angle, the equilateral triangle described upon the sum of these sides will be equal to that described upon the base, together with triple the original triangle.

E L E M E N T S

P R O P O S I T I O N S

O F

G E O M E T R Y.

B O O K II.

D E F I N I T I O N S.

1. **B**Y the rectangle under two straight lines is meant the rectangle, of which these lines (or others equal to them,) are two adjacent sides.

2. If through any point, in the diagonal of a parallelogram, two straight lines be drawn parallel to its sides, these divide the whole figure into four other parallelograms; the two through which the diagonal passes are called the parallelograms about the diagonal, and the remaining two the complements.

3. If a straight line be cut at any point (between or beyond its extremes,) the distances of that point from its extremities are called the segments of the line, and its distance from the middle of the line the mean distance of the point of section.

E

P R O P.

PROP. I. THEOR.

The opposite sides and opposite angles of a parallelogram are equal, and the diagonal divides it into two equal parts.

Fig. 31.

LET ABCD be a parallelogram, of which BD is the diagonal, the opposite sides shall be equal, AB to CD, and BC to AD; the opposite angles equal, DAB to DCB, and ADC to ABC; and the diagonal BD shall divide the parallelogram into two equal triangles DAB, DCB.

a 16. 1.

For, in these triangles, the angles ADB, DBC are equal, because they are alternate angles^a; and the angles ABD, BDC equal for the same reason; also the side BD, between the angles of each, is common to both; therefore these triangles are equal in every

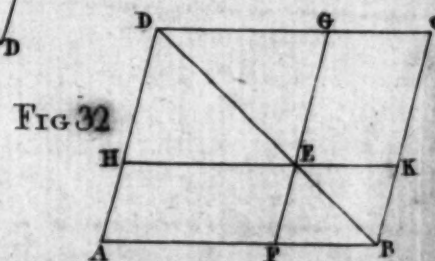
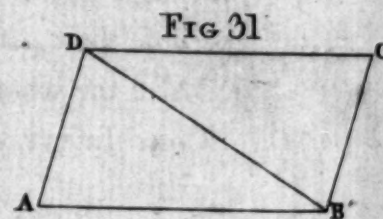
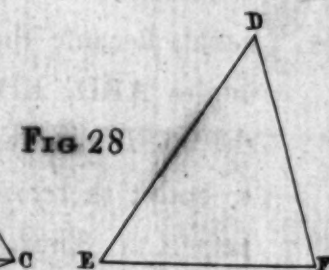
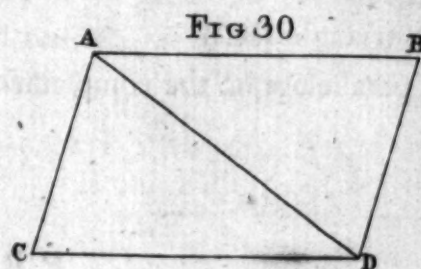
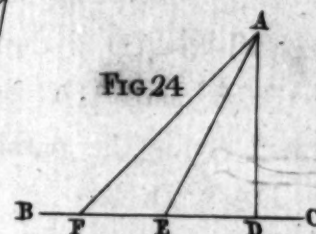
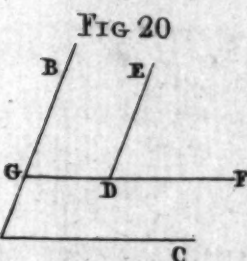
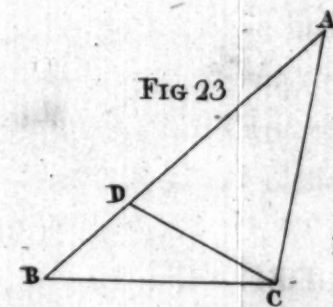
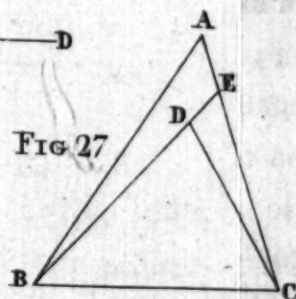
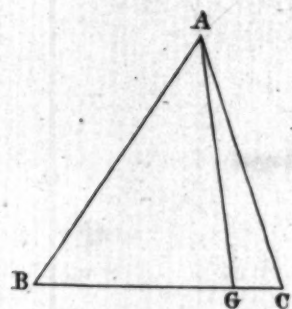
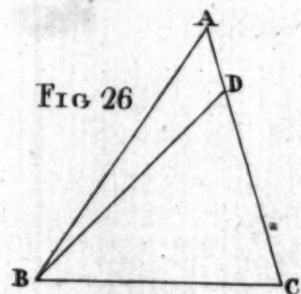
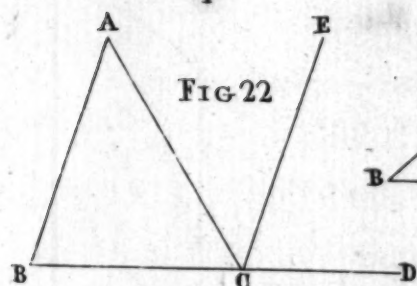
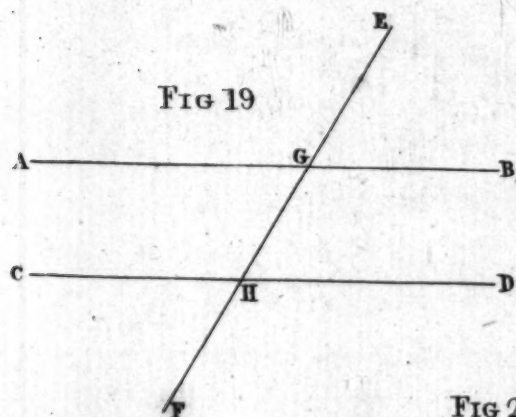
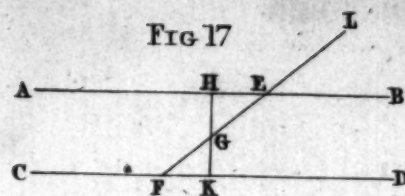
b 5. 1.

respect^b. Thus, AB is equal to CD, BC to AD, the angle DAB to DCB, the angle ADC to ABC, (because their parts are equal), and the whole triangle DBA to the whole triangle DCB.

Fig. 32.

COR. In every parallelogram the complements are equal.

PROP.



GEOMETRY

FIG 18

PL. II. pa. 34.

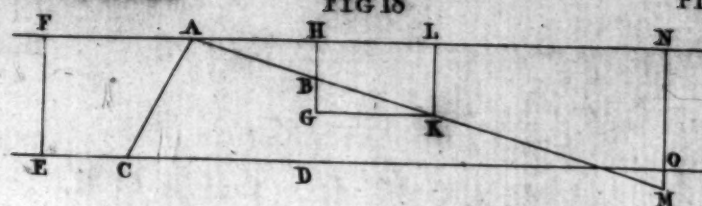
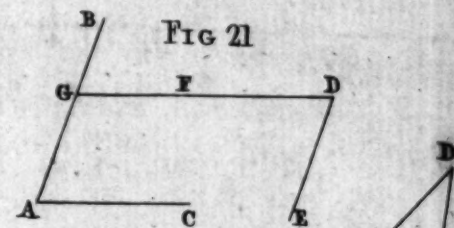


FIG 21



THE OLD CHURCH

THE OLD CHURCH

THE OLD CHURCH

THE OLD CHURCH

THE OLD CHURCH

THE OLD CHURCH

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THE OLD CHURCH

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THE OLD CHURCH

THE OLD CHURCH

PROP. II. THEOR.

Parallelograms or triangles on the same, or equal bases, and between the same parallels, are equal.

I. PARALLELOGRAMS. 1. Let there be two parallelograms AC, AF, on the same base AB, and between the same parallels DF, AB, they shall be equal to one another. For the triangles ADE, BCF, having the side AD equal to BC^a, AE to BF^a, and likewise the included angles DAE, CBF equal^b, are equal in every respect^c. Therefore, each of them being taken from the same figure ABFD, the remainders must be equal, viz. the parallelogram AC to the parallelogram AF.

Fig. 33.

2. Let AC, HF, be two parallelograms upon equal bases AB, HG, and between the same parallels AG, DF; AC, HF, shall be equal. For, let AE, BF, be drawn; then AB, being equal to HG, will be equal to EF^a; and it is also parallel to EF; hence the figure AEFB is a parallelogram^b; and it is equal to each of the parallelograms AC, HF; consequently they are equal to one another.

Fig. 34.

II. TRIANGLES. 1. Let ABC, ABE, be two triangles on the same base AB, and between the same parallels AB, CE; they shall be equal to one another. For, let AD be drawn parallel to BC^a, and BF

Fig. 35.

^a cor. 4.
^b 14. 1.

BF to AE, and let them meet CE produced in D, F; then the parallelograms DB, AF are equal; and consequently their halves, the triangles ABC, ABE, are also equal.

Fig. 36.

2. When the triangles are upon equal bases, and between the same parallels, they may be proved equal in the same manner.

COR. 1. Of triangles between the same parallels, the greater triangle has the greater base; and conversely.

COR. 2. If a parallelogram and a triangle be upon the same or equal bases, and between the same parallels, the parallelogram is double the triangle.

COR. 3. Equal triangles, upon the same base, or upon equal bases, in the same straight line, and upon the same side of it, are between the same parallels.

P R O P. III. P R O B.

To describe a square upon a given straight line.

Fig. 37.

Let AB be the given straight line, upon which it is required to describe a square. From one extremity A, let the perpendicular AE be raised^a, from which let AD be cut off equal to AB^b; and through B, D, let BC, DC be drawn parallel to AD, AB^c; then

^a 9. 1.

^b 3. 1.

^c cor. 4.
14. 1.

then the figure ABCD is a parallelogram by construction; hence AD is equal to BC^d, and AB to DC; but AD is equal to AB; therefore the four sides are equal to one another. Again, because the two interior angles A, B are together equal to two right angles^e, and one of them A is a right angle, the other B must also be a right angle. In like manner, C and D are proved to be right angles. Consequently the figure ABCD is a square^f.

COR. 1. A rectangle, under two given lines, may be formed in a similar manner.

COR. 2. A parallelogram, having one angle right, is a rectangle.

COR. 3. A parallelogram, having one angle right, and two adjacent sides equal, is a square.

COR. 4. The rectangle, under two equal lines, is a square.

PROP. IV. THEOR.

If two adjacent sides of one rectangle be equal to two adjacent sides of another, each to each, the rectangles are equal in every respect.

Let ABCD, EFGH be two rectangles, having the sides AB, BC equal to EF, FG, each to each, these rectangles are equal in every respect.

For,

Fig. 38.

ax. 1. For, if the rectangle AC were applied to EG, so that the point A should fall upon E, and the side AB upon EF, then AB, EF would coincide^a; because they are equal; AD, BC would fall upon EH, FG^a; because the angles are right; the point C would fall upon G, because BC, FG are equal; and the side CD would fall upon GH, because the angles C, G are right. Consequently the point D would fall upon H, and the rectangles coincide.

COR. 1. The squares of equal lines are equal; and, conversely, equal squares have equal sides.

COR. 2. Parallelograms, or triangles of equal bases and altitudes, are equal.

PROP. V. THEOR.

If a straight line be cut at any point, the mean distance of the point of section is equal to half the sum, or half the difference of the segments, according as the point is beyond or between the extremities of the line.

Fig. 39.

1. Let the straight line AB, of which C is the middle point, be cut at any other point D, between its extremes; then CD is equal to half the difference of AD, DB. For, since AC is equal to CB, AD is equal to CB, CD, taken together; and, from these equals, DB being taken away, the difference of AD, DB is equal to the double of CD^a.

ax. 3.

2. Let

2. Let the straight line AB, of which C is the middle point, be cut at any point D, beyond its extremes; then CD is equal to half the sum of AD, DB. For, since AC is equal to CB, AD is equal to CB, CD, taken together; and to these equals DB being added, the sum of AD, DB is equal to the double of CD. Fig. 40.

COR. 1. The greater of two unequal magnitudes is equal to half their sum added to half their difference; and the less equal to half their difference taken from half their sum.

COR. 2. If the excess of the first of three magnitudes above the second be equal to the excess of the second above the third, the second is equal to half the sum of the first and third.

PROP. VI. THEOR.

The sum of all the rectangles under one straight line, and the several parts of another, is equal to the rectangle under the two whole lines.

Let there be two straight lines, AB, and C, of which AB is divided into any number of parts, AE, EF, FB; then shall the sum of the rectangles under C and AE, C and EF, C and FB, be equal to the rectangle under C and AB. Fig. 41.

For,

For, the perpendicular AH being raised from the point A, equal to C, and the parallelogram HB completed, let the straight lines EL, FK be drawn from the several points of division in AB parallel to AH. Then the parallelogram AG, and the parts AL, EK, FG, into which it is divided by these lines, are rectangles; and, because AH is equal to C, and EL, FK, each equal to AH, AG is the rectangle under C and AB, AL under C and AE, EK under C and EF, FG under C and FB. Now, the whole AG being equal to all its parts AL, EK, FG, taken together, the proposition is manifest.

COR. 1. If a straight line be divided into any number of parts, the sum of the rectangles under the whole line, and each of the parts, is equal to the square of the whole line.

COR. 2. If a straight line be divided into any two parts, the rectangle under the whole line, and one of the parts, is equal to the square of that part, together with the rectangle under the two parts.

COR. 3. The rectangle under one straight line, and the double, triple, &c. of another, is equal to twice, thrice, &c. the rectangle under the two lines.

P R O P.

PROP. VII. THEOR.

Twice the rectangle under the segments of a straight line is equal to the difference between the square of the line, and the sum of the square of its segments.

Let AB be the straight line, and AD, DB its segments; twice the rectangle under AD, DB is equal to the difference between the square of AB and the sum of the squares of AD, DB. Fig. 42.

For, let the square ABEF be described upon AB^a, from BE or BE produced, (according as the point D is between or beyond A, B), let BG be cut off equal to BD, and through G, D, let GK, DL be drawn parallel to AB, BE, intersecting each other in H, and meeting the opposite sides of the square in K, L. Then, because AB, BE are equal, and BD, BG equal, the wholes or remainders AD, GE are equal; but AD is equal to FL^b, and GE to FK; ^a 3. 2. ^b 1. 2. therefore FK is equal to FL. Hence the figure KHLF is a square^c, and equal to the square of AD^d. ^c 3. c. 3. 2. ^d 1. c. 4. 2. And DBGH is the square of DB^e. Now the difference between the square ABEF and the sum of the squares KHLF, DBGH is the sum of the rectangles AH, HE, which is equal to twice the rectangle under AD, DB^e. ^e 4. 2.

F

COR.

COR. 1. If a straight line be divided into two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle under these parts.

COR. 2. The square of a straight line is equal to four times the square of half the line; and half its square is double the square of its half.

COR. 3. Twice the rectangle, under two unequal straight lines, is equal to the difference between the square of their sum and the sum of their squares.

COR. 4. Twice the rectangle, under two unequal straight lines, is equal to the difference between the sum of their squares and the square of their difference.

COR. 5. Of two unequal straight lines the excess of the square of their sum above the sum of their squares, is equal to the excess of the sum of their squares above the square of their difference.

COR. 6. Of two unequal straight lines, the sum of their squares is equal to double the square of half their sum added to double the square of half their difference^a.

^a cor. 5.
cor. 2. 5.
2. & cor. 2.

COR. 7. If a straight line be cut at any point, the sum of the squares of its segments is double the sum of the squares of half the line, and the mean distance of the point of section.

P R O P.

PROP. VIII. THEOR.

The rectangle under the sum and difference of two unequal straight lines, is equal to the difference of their squares.

Let AB, CD, be two unequal straight lines, of which AB is the greater, the rectangle under the sum and difference of AB, CD is equal to the difference of their squares. For, let the square ABEF be described upon AB; from EF, EB, and AB produced, let EH, EK, and BG, be cut off, each equal to CD, and let KL, parallel to AB, intersect GM, HN, parallel to BE, in M, N, and AF in L. The figure EKNH is a square^a, and equal to the square of CD^b; therefore the sum of the rectangles HL, LB is the difference of the squares of AB, CD. Again, because EB is equal to AB, and EK to CD, BK or GM, which is equal to it, is the difference of AB, CD, and AG is their sum; therefore AM is the rectangle under their sum and difference.* Now, the rectangle AM is equal to the sum of the rectangles HL, LB, because LB is common to both, and the remaining rectangles HL and BM are contained by equal lines.

^a 3. c. 3.

^{2.}

^b 1. c. 4.

^{2.}

COR. The rectangle under the segments of a straight line is equal to the difference between the squares

squares of the half line, and the mean distance of the point of section ; and the sum of the rectangle, and one of the squares, is equal to the other.

P R O P. IX. T H E O R.

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Fig. 44

Let ABC be a right angled triangle, of which AC is the hypotenuse ; the square of AC is equal to the sum of the squares of AB, BC.

* 3. 2.

For, let the squares AE, FB, BI be described upon the three sides * ; through B let LM be drawn parallel to AD, meeting FG in M, and DE in L ; also, let DA be produced to meet FG in K. The triangles

° 5. 1.

AFK, ABC are equal in every respect ° ; for the angles at F and B are equal, because they are right angles ; the angles FAK, BAC are equal, because each of them is less than a right angle by the same angle KAB ; and the sides AF, AB are equal, because they are sides of the same square. Thus, AC or AD is equal to AK, and therefore the parallelogram AL equal to the parallelogram AM. But AM is equal to FB. Consequently the parallelogram AL is equal to the square FB. In the same manner,

Fig 33

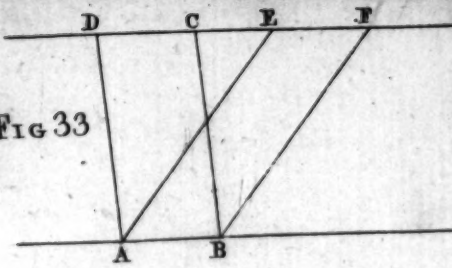


Fig 34

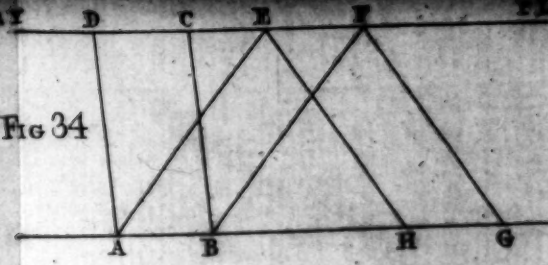


Fig 35

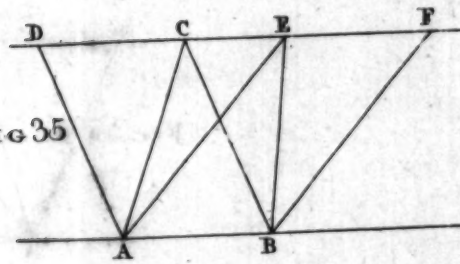


Fig 36

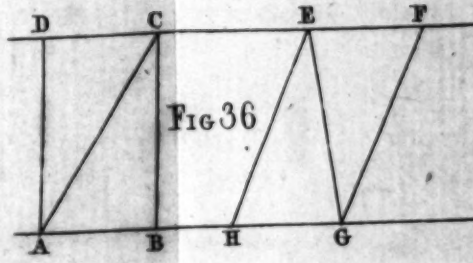


Fig 37



Fig 39

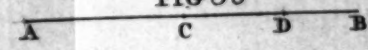


Fig 38

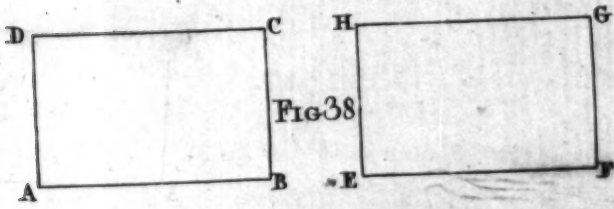


Fig 40

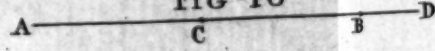


Fig 41

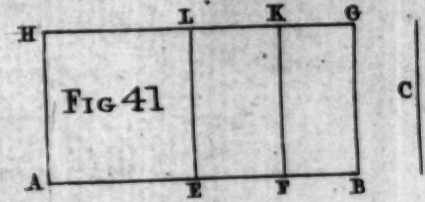


Fig 42

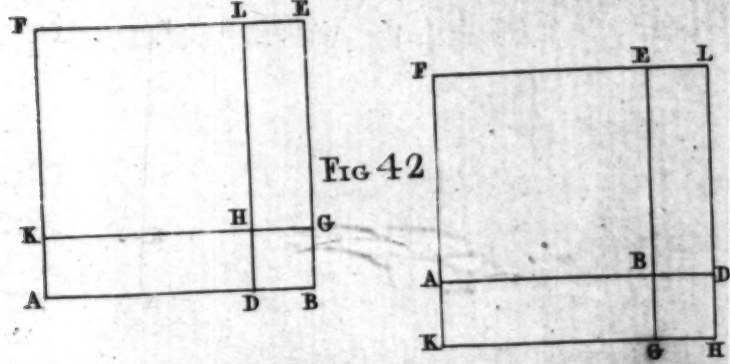


Fig 43

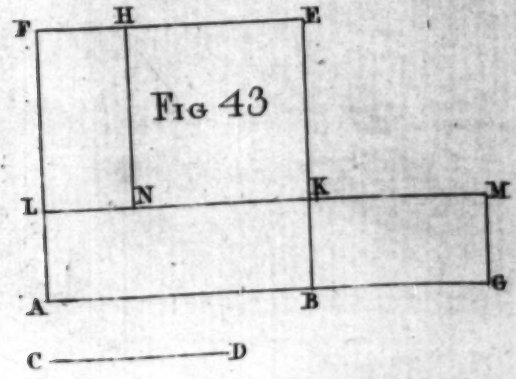
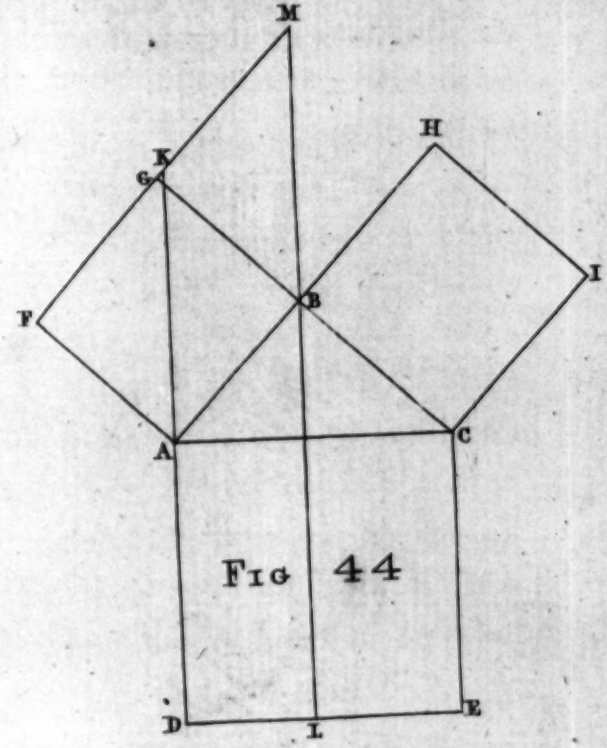


Fig 44



manner, the parallelogram CL may be proved equal to the square BI. Therefore the whole square AE is equal to the sum of the squares FB, BI.

COR. The square of one of the sides of a right angled triangle is equal to the difference of the squares, or the rectangle under the sum and difference, of the hypotenuse and the other side.

PROP. X. THEOR.

In every triangle, the square of one of the sides is greater or less, according as the opposite angle is obtuse or acute, than the sum of the squares of the other side and the base, by twice the rectangle under the base and the distance of the perpendicular from the angular point.

Let ABC be a triangle, having an obtuse or an acute angle at B, and let CD be the perpendicular from the vertex C upon the base AB; the square of AC is greater or less than the sum of the squares of AB, BC, by twice the rectangle under AB, BD. Fig. 45.

For, since the square of AC is equal to the sum of the squares of AD, DC², and the square of BC equal to the sum of the squares of BD, DC², the difference between the square of AC, and the sum of the squares of AB, BC, is equal to the difference between

between the sum of the squares of AD, DC, and the sum of the squares of AB, BD, DC; which is the same with the difference between the square of AD and the sum of the squares of AB, BD^b. But the square of AD is greater or less than the sum of the squares of AB, BD, by twice the rectangle under AB, BD, according as the point D is on the other side of B, or on the same side with A^c, that is, according as the angle BCA is obtuse or acute^d. Therefore also the square of AC is accordingly greater or less than the sum of the squares of AB, BC, by twice the rectangle under AB, BD.

In the case in which the point D coincides with A, CAB is a right angle, and consequently CBA an acute angle; and, since the square of AC is less than the square of BC, by the square of AB^e, it will be less than the sum of the squares of AB, BC, by twice the square of AB, which is the same with twice the rectangle under AB, BD^f.

COR. 1. If a straight line be drawn from the vertex of a triangle to the middle of the base, the sum of the squares of that line, and half the base, is equal to half the sum of the squares of the two sides^g.

COR. 2. The vertical angle of a triangle is obtuse, right, or acute, according as the square of the base is greater,

greater, equal, or less, than the sum of the squares of the two sides.

PROP. XI. THEOR.

The difference of the squares, or the rectangle under the sum and difference, of the two sides of a triangle, is equal to twice the rectangle under the base, and the distance of the perpendicular from the middle of the base.

Let ABC be any triangle, of which the base BC is Fig. 46.
bisected in E, and let AD be the perpendicular; the difference of the squares of BA, AC is equal to twice the rectangle under BC and ED.

For, since the square of BA is equal to the squares of BD, DA^a, and the square of AC equal to the squares of DC, DA^a; the difference of the squares of BA, AC is equal to the difference of the squares of BD, DC^b. But the difference of the squares of BD, DC, is equal to the rectangle under their sum and difference^c, that is, to the rectangle under BC, and twice ED^d, or twice the rectangle under BC and ED^e. Therefore the difference of the squares of BA, AC, or the rectangle under their sum and difference, is equal to twice the rectangle under BC and ED.

In

In the case in which the point D coincides with B or C, the proposition is evident.

PROP. XII. PROB.

To divide a given straight line into two parts, so that the rectangle under the whole line, and one of the parts, may be equal to the square of the other part.

Fig. 47.

Upon the given straight line AB let the square AC be described^a; let the side AD be bisected in E, and the points E, B joined; in EA produced let there be taken EF equal to EB, and let the square AFHG be described upon AF: The line AB will be divided in the point G, so that the rectangle under AB, BG shall be equal to the square of AG. For, let HG be produced to meet DC in K; the rectangle under DF, FA, together with the square of AE, is equal to the square of EF^b, or EB. But the square of EB is equal to the squares of AB, AE^c. Therefore the rectangle under DF, FA, together with the square of AE, is equal to the squares of AB, AE; and the rectangle under DF, FA, equal to the square of AB, that is, DH equal to AC; from which the common part AK being taken away, there remains FG equal to GC, that is, the square of AG, equal to the rectangle under AB, BG.

COR.

COR. The line that is compounded of the given line, and its greater segment, is also divided in the same manner.

A P P E N D I X.

PROP. 1. Every quadrilateral figure, whose opposite sides or opposite angles are equal, is a parallelogram.

PROP. 2. If two sides of a triangle be bisected, the straight line which joins the points of section will be parallel to the remaining side, and equal to the half of it.

PROP. 3. To make a triangle that shall be equal to a given quadrilateral figure.

PROP. 4. If, from any point, in an equilateral triangle, perpendiculars be let fall upon the sides, these perpendiculars, taken together, shall be equal to the perpendicular of the triangle.

PROP. 5. If a straight line be drawn from the vertex, to any point in the base of an isosceles triangle, the square of that line, together with the rectangle under the segments of the base, will be equal to the square of one of the equal sides of the triangle.

PROP. 6. To find the side of a square that shall be equal to two or more given squares.

G

PROP.

PROP. 7. If, from the vertex of a triangle, a straight line be drawn to bisect the base, it will also bisect any other straight line parallel to the base, and terminating in the two sides of the triangle.

PROP. 8. The two diagonals of a parallelogram mutually bisect each other, and the sum of their squares is equal to the sum of the squares of all the sides.

PROP. 9. To make a rectangle that shall be equal to a given rectilineal figure.

PROP. 10. If, from any point, straight lines be drawn to the four angular points of a rectangle, the sums of the squares of those drawn to the opposite angles will be equal,

PROP. 11. In any quadrilateral figure, the sum of the squares of the diagonals, and four times the square of the line joining their middle points, is equal to the sum of the squares of all the sides,

PROP. 12. To find the side of a square that shall be equal to the difference between two given squares.

PROP. 13. In any quadrilateral figure, the sum of the squares of the diagonals is equal to double the sum of the squares of the two lines that join the middle points of its opposite sides.

PROP. 14. If the vertical angle of a triangle be two thirds of two right angles, the square of the base will be equal to the rectangle under the two sides added to the sum of their squares : But, if it be two thirds

thirds of one right angle, the square of the base will be equal to the difference of the rectangle under the two sides, and the sum of their squares.

PROP. 15. To bisect a given triangle by a straight line drawn from a given point in one of its sides.

PROP. 16. If, from the extremities of the base of a triangle, straight lines be drawn to bisect the opposite sides, another straight line drawn through their intersection, from the vertex, will bisect the base.

PROP. 17. If a straight line be drawn parallel to the base of a triangle, and the diagonals of the quadrilateral figure be drawn, then a straight line, drawn through their intersection, from the vertex, will bisect the base.

PROP. 18. To find the side of a square that shall be equal to a given rectangle.

PROP. 19. The equilateral triangle, described upon the hypotenuse of a right angled triangle, is equal to the sum of the equilateral triangles described upon the two sides.

PROP. 20. If, through one of the angular points of a square, a straight line be drawn, to cut the two opposite sides, and from one of the points of section a perpendicular to that straight line be raised to meet the side opposite to that from which it is drawn; then shall the square of the produced part of that opposite

posite side be equal to the square of the line between the points of section added to the original square.

PROP. 21. To divide a given straight line into any proposed number of equal parts.

PROP. 22. If a straight line be divided into three parts, so that the squares of the lines, compounded of two adjacent parts, be together equal to the square of the whole line, then shall the rectangle under the two extreme parts be equal to double the rectangle under the whole line, and the middle part; and conversely.

PROP. 23. If a parallelogram be constituted upon the base of a triangle, and on the same side with the triangle, and, upon the other two sides, parallelograms be likewise constituted, so that their opposite sides pass through the angular points of the former; then shall the parallelogram on the base be equal to the sum or difference of those on the sides, according as they fall both without the triangle, or one of them, upon it.

PROP. 24. Let there be four points, A, B, C, D, in the same straight line, and let E be the point in that line that is equally distant from the middle of the segments AB, CD; then, F being any other point whatever, the sum of the squares of the distances of F from the four points A, B, C, D, shall exceed the sum of the squares of the distances of E from the same points, by four times the square of EF.

ELEMENTS

OF

GEOMETRY:

BOOK III.

DEFINITIONS.

1. **A** Diameter of a circle is a straight line drawn through the center, and terminated on both sides by the circumference. It divides the circle into two parts, called semicircles.

2. An arch of a circle is any part of its circumference.

3. The chord of an arch is the line joining its extreme points.

4. A segment of a circle is a figure bounded by an arch and its chord.

5. A sector of a circle is a figure bounded by two radii, and the arch between them: When the radii are perpendicular to one another, it is called a quadrant.

6. An angle in a segment is that which is contained by two straight lines, drawn from any point in its arch, to the extremities of its chord.

7. An

7. An angle is said to stand upon the arch that is intercepted between its sides.

8. A tangent to a circle is a straight line, which meets the circumference in one point, and every where else falls without it.

9. Two circles are said to touch one another when the circumference of one of them meets the circumference of the other in one point, and every where else falls without it.

10. A straight line is said to be applied in a circle when both its extremes are in the circumference.

11. When all the angular points of a rectilineal figure are in the circumference of a circle, the figure is said to be inscribed in the circle, or the circle described about the rectilineal figure.

12. When all the sides of a rectilineal figure touch the circumference of a circle, the figure is said to be described about the circle, or the circle inscribed in the rectilineal figure.

13. When all the angular points of a rectilineal figure are placed in the sides of another, and the one figure wholly included in the other, then the former is said to be inscribed in the latter, or the latter described about the former.

P R O P.

PROP. I. PROB.

To find the center of a given circle.

Let ADB be the given circle, of which it is required Fig. 48.
to find the center.

Having assumed two points, A, B, in the circumference, let the straight line AB which joins them be bisected at right angles by the line CD^a meeting the circumference in D, E, then DE being bisected in F, the point F will be the center of the circle. ^a 8. 1.

For, if not, let any other point G, without the line DE, be the center, and let GA, GB, GC, be drawn. The triangles GAC, GBC have the sides GA, GB equal, as being radii of the circle, also AC, CB equal by construction, and CG common; therefore they are equal in every respect^b. Hence ^b 6. 1. the angles GCB, GCA are equal, and each of them a right angle. But FCB is a right angle: Therefore GCB must be equal to FCB, a part to the whole, which is absurd.

COR. 1. A straight line, which bisects a chord at right angles, passes through the center of the circle.

COR. 2. If two chords be bisected by perpendiculars, their intersection will be the center: And thus the center of a given segment may be determined.

PROP.

PROP. II. THEOR.

If a straight line, drawn through the center of a circle, bisect any chord of the same, not passing through the center, it will also bisect the arches which the chord subtends.

Fig. 49.

In the circle AFB, let the straight line FCE, passing through the center C, bisect the chord AB in D, and it shall also bisect the arches AFB, AEB.

For, let AC, CB, be drawn; then the triangles ADC, BDC, having AC, CB equal, as being radii of the circle, AD, DB equal by hypothesis, and CD common to both, are equal in every respect^a. Hence the angle ACE is equal to BCE. Now, if the semicircle FAE, by revolving upon FE, were applied to the semicircle FBE, the line AC would fall upon CB, because the angles ACE, BCE are equal^b, the point A would fall upon B, because AC, CB are equal, and every point in the arch FA would fall upon FB, because every point in each of them is equally distant from C, that is, the arches FA, FB would coincide; consequently they are equal.

ax. 1.

COR. I. If a straight line do any two of these five, pass through the center of a circle, bisect an angle at the center, bisect the arch upon which it stands, bisect the chord of that arch, cut the chord at right angles;

angles ; it must necessarily do the other three, provided the chord be not a diameter in that case which is stated in the proposition.

COR. 2. If a straight line pass through the center of a circle, it bisects the circle, and its circumference ; and conversely.

COR. 3. In the same circle, or in circles of equal radii, equal angles at the center stand upon equal arches, and the greater angle stands upon the greater arch ; and conversely.

COR. 4. A right angle at the center of a circle stands upon a fourth part of the circumference ; and conversely.

COR. 5. Whatever part an angle at the center is of four right angles, the same part will the arch upon which it stands be of the whole circumference.

COR. 6. When the radii are equal, sectors which have equal angles are equal ; also semicircles are equal, and circles equal.

COR. 7. The method of bisecting a given arch, or a given segment, is evident.

COR. 8. A straight line, bisecting two parallel chords, passes through the center.

G

PROP.

PROP. III. THEOR.

In the same semicircle, or in semicircles of equal radii, equal chords subtend equal arches, the greater chord subtends the greater arch; and conversely.

Fig. 50.

Let AHB, DKE be two semicircles of equal radii AC, DF, and in these let AH, DG be two equal chords; the arches which they subtend will be equal. For the radii CH, FG being drawn, the triangles ACH, DFG have all their sides equal, each to each; therefore the angle HCA is equal to GFD^a, and consequently the arches upon which they stand are equal^b; also conversely, if the arches be equal, their chords will be equal: For, since the arches AH, DG are equal, the angles ACH, DFG are equal^b. But the sides AC, CH are equal to DF, FG. Therefore, in the triangles ACH, DFG, the base AH is equal to the base DG^c.

Next, let the chord DK be greater than the chord AH, and the arch DK will also be greater than the arch AH. For the triangles DFK, ACH have the sides DF, FK equal to AC, CH, but the base DK greater than the base AH: Therefore the angle DFK is greater than the angle ACH^d, and consequently the arch DK greater than AH. The converse is proved in like manner.

PROP.

PROP. IV. THEOR.

The diameter is the greatest chord in a circle; those chords which are equally distant from the center are equal; and that which is nearer the center is greater than that which is more remote; and conversely.

Let AFE be a circle, of which AB is a diameter, Fig. 51. AB is greater than any other chord, as DE, that does not pass through the center. For DC, EC being drawn to the center, AB is equal to the sum of DC, CE, and therefore greater than DE.^a

^a 20. 1.

Next, let AF, DE be two chords equally distant from the center, that is, let the perpendiculars CG, CH be equal, then AF is equal to DE. For the right angled triangles AGC, DHC having their hypotenuses AC, CD equal, and the sides CG, CH equal, are equal in every respect^b. Thus, AG is equal to DH, and their doubles^c AF, DE also equal.

^b cor. 24.

^{1.}

^c cor. 1.

^{2.} 3.

Lastly, Let DE be nearer to the center than AK, that is, let the perpendicular CH be less than CL, then DE is greater than AK. For the sum of the squares of CH, HD is equal to the sum of the squares of CL, LA, because each of them is equal to the square of the radius^d. And from these the unequal squares of CH, CL being taken away, there remains

^d 9. 2.

the

the square of HD greater than the square of LA ; therefore the line HD is greater than LA, and DE, the double of HD, greater than AK, the double of LA.

PROP. V. THEOR.

Of all the straight lines that can be drawn to meet the circumference of a circle, from any point that is not the center, that which passes through the center is the greatest ; that which, when produced, passes through the center, is the least ; those which intercept equal arches, from the extremity of the greatest, are equal, and that which intercepts the less arch is the greater ; and conversely.

Fig. 52.

Let C be the center of the circle, and A any other point. Of all the lines that can be drawn from A to the circumference, AB, which passes through the center, is the greatest. For, if any other, as AE, be drawn ; then, CE being joined, AB is equal to the sum of AC, CE, and therefore greater than AE^a.

^a 20. 1.

Next, AD, which, when produced, passes through the center, is the least. For, if any other, as AG, be drawn, and CG joined ; then CD, being equal to CG, will be less than the sum of CA, AG^a, and therefore AD less than AG.

Further,

Further, AE, AF, which intercept equal arches from the point B, are equal. For, since EB, BF are equal, the angles ECB, BCF^b, and consequently ACE, ACF^c, are also equal. But, in the triangles ACE, ACF, the sides EC, CF are equal, and AC common to both; therefore the base AE is equal to the base AF^d.

^b cor. 3.

^c 2. 3.

^c 12. 1.

^d 4. 1.

Lastly, AE, which intercepts the less arch, is greater than AG. For, since the arch EB is less than GB, the angle ECB will be less than GCB^b, and consequently ACE greater than ACG^c. But, in the triangles ACE, ACG, the sides CE, CG are equal, and AC common to both; therefore the base AE is greater than the base AG^c.

^c 23. 1.

COR. 1. There cannot be drawn more than two equal straight lines, to the circumference of a circle, from any point that is not the center.

COR. 2. If there be three equal straight lines drawn from a certain point, to the circumference of a circle, that point is the center.

PROP. VI. THEOR.

An angle at the center of a circle is double the angle at the circumference which stands upon the same arch.

Let

Fig. 54. Let ACB be an angle at the center, and ADB an angle at the circumference, upon the same arch AB , the angle ACB is the double of ADB .

Fig. 53. CASE 1. When the center is in DB , one of the lines that contain the angle at the circumference: The triangle ACD is isosceles; therefore the angles at A and D are equal^a, and the angle ACB , which is equal to both^b, will be double one of them.

^a cor. 4. 1.

^b 18. 1.

Fig. 54. CASE 2. When the center is between the lines AD , DB , let the diameter DCE be drawn. The angle ACE is the double of ADE , and the angle ECB the double of EDB ; therefore the whole angle ACB is double the whole angle ADB .

Fig. 55. CASE 3. When the center is without the lines AD , DB ; then, as before, the angle ECB is the double of EDB , and the part ACE the double of ADE ; therefore the remaining angle ACB is double the remaining angle ADB .

P R O P. VII. T H E O R.

All angles in the same segment of a circle are equal.

Fig. 56. Let $ADEB$ be any segment of a circle, all the angles in that segment are equal. For, from the center C , let CF be drawn perpendicular to AB ^a, meeting the circumference of the other segment AGB in G ,

^a 11. 1.

PROPOSITION 1

Let ABC be a triangle. I say that the angles ABC and ACB are together less than two right angles.

For, if they were equal to two right angles, the lines AB and AC would be parallel, and the triangle ABC would be impossible.

Therefore the angles ABC and ACB are together less than two right angles.

Q.E.D.

PROPOSITION 2

Let ABC be a triangle. I say that the angles ABC and ACB are together less than two right angles.

For, if they were equal to two right angles, the lines AB and AC would be parallel, and the triangle ABC would be impossible.

Fig. 54. Let ACB be an angle at the center, and ADB an angle at the circumference, upon the same arch AB , the angle ACB is the double of ADB .

Fig. 53. CASE 1. When the center is in DB , one of the lines that contain the angle at the circumference: The triangle ACD is isosceles; therefore the angles at A and D are equal^a, and the angle ACB , which is equal to both^b, will be double one of them.

^a cor. 4. 1.

^b 18. 1.

Fig. 54. CASE 2. When the center is between the lines AD , DB , let the diameter DCE be drawn. The angle ACE is the double of ADE , and the angle ECB the double of EDB ; therefore the whole angle ACB is double the whole angle ADB .

Fig. 55. CASE 3. When the center is without the lines AD , DB ; then, as before, the angle ECB is the double of EDB , and the part ACE the double of ADE ; therefore the remaining angle ACB is double the remaining angle ADB .

PROP. VII. THEOR.

All angles in the same segment of a circle are equal.

Fig. 56. Let $ADEB$ be any segment of a circle, all the angles in that segment are equal. For, from the center C , let CF be drawn perpendicular to AB ^a, meeting the circumference of the other segment AGB in G ,

^a 11. 1.

FIG. 45.

GEOMETRY

FIG. 48.

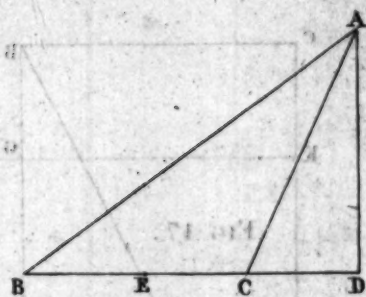
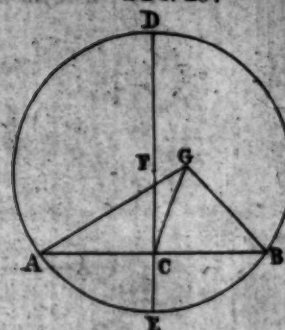
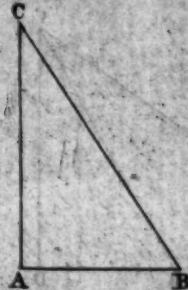
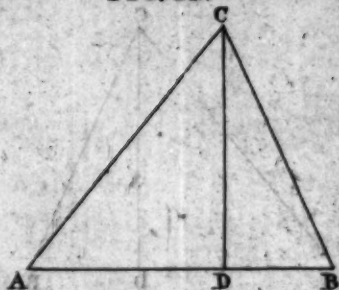
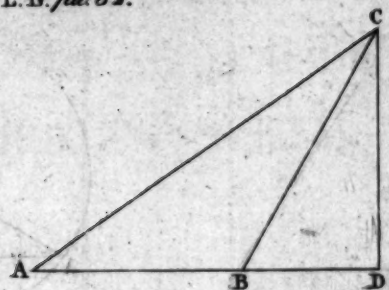


FIG. 46.

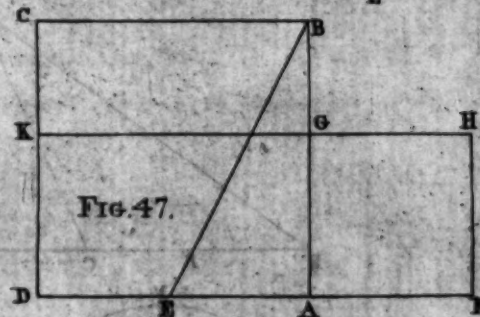
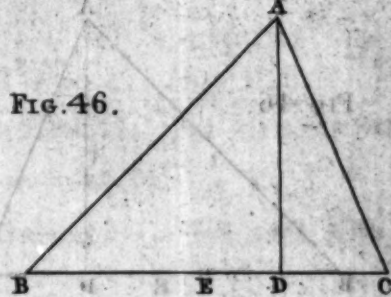


FIG. 47.

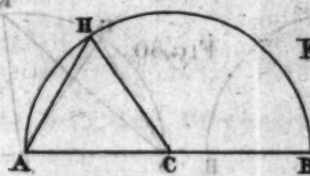
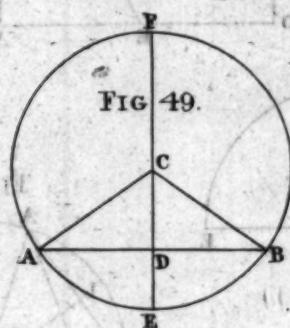


FIG. 50.

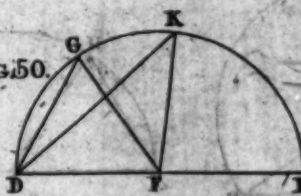


FIG. 51.

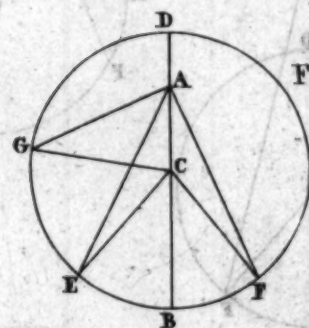
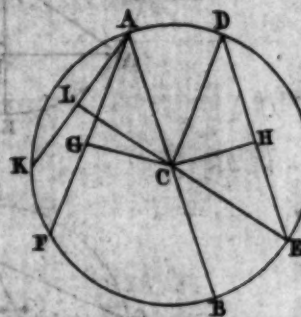


FIG. 52.

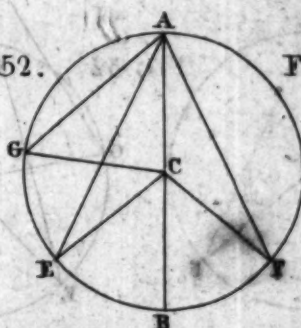


FIG. 52.

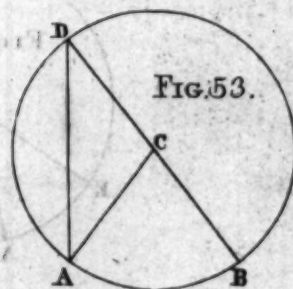
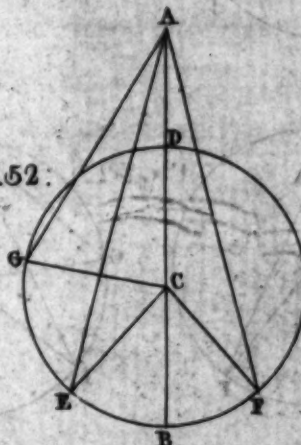


FIG. 53.

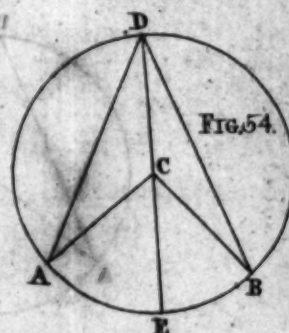


FIG. 54.

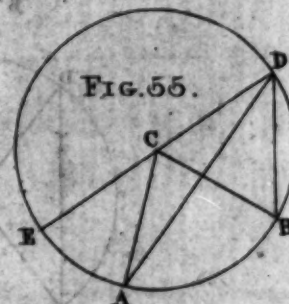
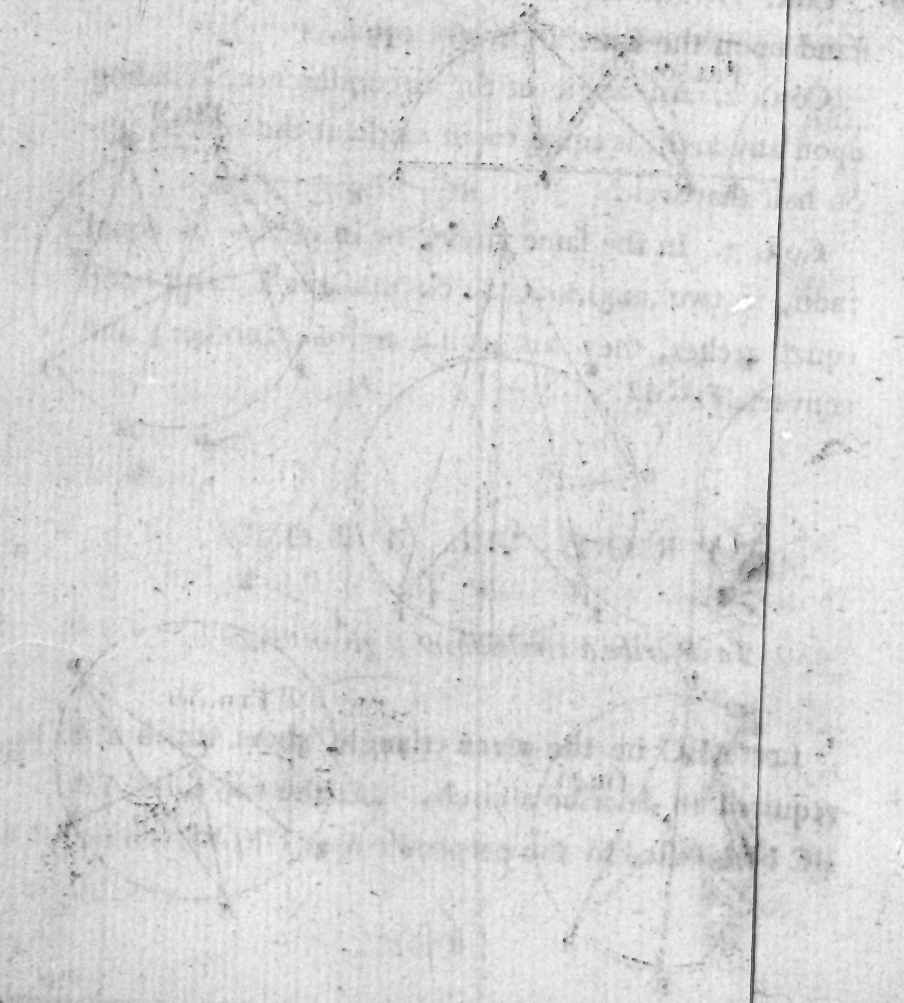


FIG. 55.

BOOK III
 CHAPTER I
 OF THE NATURE OF THE
 HUMAN MIND
 IN THE STATE OF NATURE
 The first thing that strikes the eye
 of a philosopher, when he considers
 the human mind, is the vastness
 of its capacity. It is able to receive
 an infinite number of ideas, and to
 combine them in an infinite number
 of ways. This is the source of all
 our knowledge, and the foundation
 of all our reasoning.



G, and let AC, CB and DG be joined. Then, because CF passes through the center, and is perpendicular to AB, it bisects the arch AGB^b. Hence the angles ACG, BCG are equal^c. But the former is the double of the angle ADG^d, and the latter the double of BDG. Therefore the angle ADG is equal to BDG; and the whole angle ADB equal to ACG, or BCG. Thus, the angle ACG is equal to any angle in the segment ADEB. Consequently all the angles in that segment are equal.

^b cor. 1.
^{2.} 3.
^c cor. 3.
^{2.} 3.
^d 6. 3.

COR. 1. All angles at the circumference, which stand upon the same arch, are equal.

COR. 2. An angle at the circumference, standing upon any arch, is equal to an angle at the center, upon half that arch.

COR. 3. In the same circle, or in circles of equal radii, if two angles at the circumference stand upon equal arches, they are equal to one another; and conversely.

PROP. VIII. PROB.

To describe a circle about a given triangle.

Let ABC be the given triangle, about which it is required to describe a circle. Let the two sides AB, BC be bisected by the perpendiculars DE, FE^a, intersecting

Fig. 57.

^a 8. 1.

secting each other in E, and E is the center of the circle that will circumscribe the triangle. For EA, EB, EC being drawn, the triangles ADE, BDE are equal in every respect, since AD, DB are equal, DE common to both, and the angles at D right: therefore EA is equal to EB. In the same manner, EB is proved equal to EC. If, therefore, a circle be described about the center E, at the distance EA, it will pass through the three angular points A, B, C, and circumscribe the triangle.

COR. The circumference of a circle may be described through any three points not in the same straight line.

PROP. IX. THEOR.

The angle in a semicircle is a right angle; the angle in a segment, greater than a semicircle, is an acute angle; and the angle in a segment, less than a semicircle, is an obtuse angle.

Fig. 56.

Let ADEB be any segment of the circle AEG, having the line AB for its base. From the center C let CG be drawn perpendicular to AB, meeting the circumference of the other segment AGB in G. Then, according as ADEB is a semicircle, a greater segment, or a less segment, the center C will fall upon, above, or below the line AB; and therefore the angle

angle ACG will accordingly be right, acute, or obtuse. But the angle ACG is equal to any angle, as ADB, in the segment ADEB^a. Consequently the angle ADB is right, acute, or obtuse, according as ADEB is a semicircle, a greater segment, or a less segment.

COR. 1. The center of the circumscribing circle falls within the triangle, upon one of its sides, or without, according as it is an acute-angled, right-angled, or obtuse-angled triangle.

COR. 2. In every right-angled triangle, the distance of the middle of the hypotenuse from the right-angle is equal to half the hypotenuse.

COR. 3. The angle at the vertex of any triangle inscribed in a circle, differs from a right angle by the angle which its base forms with the radius or diameter at one extremity of the base.

PROP. X. THEOR.

The opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to two right angles; and conversely.

Let ABCD be any quadrilateral figure inscribed in the circle AED, the two opposite angles at B and D are together equal to two right angles. For, let the diagonal

I

AC

Fig. 58.

AC be drawn and bisected by the perpendicular EF, meeting the circumference in E, F. Then EF passes through G, the center of the circle, and bisects the arches ABC, ADC^a. Hence the angle AGF is equal to the angle at B, and AGE to the angle at D^b. Consequently the sum of the angles at B and D is equal to the sum of the angles AGF, AGE, which is two right angles.

Fig. 59.

Also conversely, if the opposite angles at B and D, of a quadrilateral ABCD, be together equal to two right angles, a circle may be described about it. For the circumference of a circle may be described through the three points A, B, C^c; and, if that do not pass through the point D, let it meet the diagonal DB in any other point, as H. Then AH, HC being drawn, the angles at B and H would be together equal to two right angles, and therefore equal to the angles at B and D. Consequently the angle at H would be equal to the angle at D, which is impossible^d.

^d 21. 1.

COR. 1. If one of the sides of a quadrilateral figure, inscribed in a circle, be produced, the exterior angle will be equal to the interior opposite angle.

COR. 2. If, when the diagonals of a quadrilateral figure are drawn, the angles which stand upon one of the sides be equal; the circumference of a circle may be described through the four angular points.

P R O P.

PROP. XI. THEOR.

If a straight line, passing through any point in the circumference of a circle be perpendicular to the radius at that point, it will be a tangent to the circle ; and conversely.

Let the straight line DBE, passing through any point B, in the circumference of the circle AFB, be perpendicular to the radius CB ; DBE is a tangent to the circle. For, let any point D be assumed in it, from which let DC be drawn to the center. Then CD is greater than CB *, and consequently D a point without the circle. Fig. 69.
3. c. 19.
1.

Also conversely, If DE be a tangent at B, it is perpendicular to CB. For, since every point in DE, except B, is without the circle, CB is the shortest line that can be drawn to it from the center, and is therefore perpendicular to DE *.

COR. 1. At the same point, in the circumference of a circle, there cannot be more than one tangent.

COR. 2. A perpendicular to a tangent, from the point of contact, passes through the center.

COR. 3. A straight line cannot meet the circumference of a circle in more than one point, without cutting it ; or, between the circle and its tangent, a straight

straight line cannot be drawn from the point of contact.

COR. 4. A straight line cutting the circle, and parallel to a tangent, intercepts equal arches from the point of contact.

COR. 5. If two tangents pass through the extremities of a diameter, they are parallel; and conversely.

COR. 6. Parallel lines, meeting the circumference of a circle, intersect equal arches; and conversely.

COR. 7. A straight line, drawn through the middle of an arch, parallel to its chord, is a tangent to the circle.

PROP. XII. THEOR.

Two circles, whose circumferences pass through the same point, in the line passing through their centers, touch one another.

Fig. 61.

Let the circumferences of two circles, whose centers are A, B, pass through the same point C, in the straight line AB; the circles shall touch one another in that point. For, if possible, let them meet in any other point, as D, and let AD, BD be drawn: Then, because AD is equal to AC, and BD to BC; the sum or difference of AD, DB, two sides of a triangle, would be equal to the third side AB, which is absurd.

COR.

COR. 1. Two circles, whose circumferences pass through the same point, without the line, joining their centers, cut one another.

COR. 2. The circumferences of two circles cannot meet in more than one point without cutting.

COR. 3. If two circles touch one another, the straight line which joins their centers will pass through the point of contact.

P R O P. XIII. P R O B.

To draw a tangent to a given circle, from a given point without it.

Let EBC be the given circle, and A the given point, from which it is required to draw a tangent. Let AC be drawn to the center, and upon AC, as a diameter, let a circle be described, intersecting the given circle in the two points B, E. Then a straight line drawn from A, to either of these points, B, E will be a tangent to the circle EBC. For, AB, BC being drawn, because the angle in a semicircle is a right angle^a, AB is perpendicular to BC, and because AB is perpendicular to the radius BC, AB is a tangent to the circle^b.

Fig. 62.

COR. 1. From the same point, without a circle, there cannot be drawn more than two tangents.

COR.

COR. 2. Two tangents, drawn from the same point, without a circle, are equal.

PROP. XIV. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn to cut the circle, the angles which the secant makes with the tangent, are equal to those in the alternate segments; and conversely.

Fig. 63.

Let CD be a tangent to the circle ABE, and BA any secant drawn from the point of contact, the angle ABD shall be equal to any angle in the segment AEB, on the other side of AB, and ABC equal to any angle, as F, in the segment AFB. If BA be perpendicular to CD, it divides the circle into two semi-circles, each of which contains a right angle, and the angles at B are also right angles. If not, let AE be drawn parallel to CD, to meet the circumference in E, and let EB be joined. The angle ABD is equal to BAE, because they are alternate angles^a, and BAE is equal to AFB^b, because they stand upon equal^c arches; therefore the angle ABD is equal to AEB.

^a 16. 1.

^b cor. 3. 7.

^c 3.

^a cor. 4.

11. 3.

Again, because EAFB is a quadrilateral figure, inscribed in the circle, the angles at E and F are together equal to two right angles^d, and therefore equal to

^a 10. 3.

the

the two angles ABD , ABC° . From these equals let the equal angles E and ABD be taken away, and there remains the angle at F equal to ABC .

Conversely, If the secant AB make, with the straight line CD , meeting the circumference in B , the angle ABD , equal to any angle in the alternate segment, CD will be a tangent to the circle. For, the same construction being made, since the angles BAE , AEB are each equal to ABD , they are equal to one another; therefore the arch EB is equal to BA^b , and consequently CD a tangent f .

f cor. 7.
11. 3.

PROP. XV. PROB.

Upon a given straight line, to describe a segment of a circle that shall contain a given angle.

Let AB be the given straight line, and C the given angle, it is required to describe upon AB a segment of a circle that shall contain an angle equal to C . If C be a right angle, let a semicircle be described upon AB . If not, let the angle ABD be made equal to C^a , and BF drawn at right angles to BD , to meet the perpendicular EF , that bisects AB in the point F : About the center F , at the distance FB , let a circle be described. Its circumference will pass through

Fig. 64.

a 13. 1.

through the point A, because FA and FB are equal, being the bases of the triangles AEF, BEF, which are equal in every respect^b; and BD will be a tangent to that circle^c, because it is perpendicular to the radius FB. Hence the angle ABD, and consequently C, is equal to the angle in the alternate segment described upon AB^d.

PROP. XVI. THEOR.

Any angle, whose sides meet the circumference of a circle, is equal to an angle at the center of that circle, standing upon half the sum, or half the difference of the intercepted arches, according as the angular point is within or without the circle.

Fig. 65.

Let the sides AB, AC, of the angle BAC, meet the circumference of the circle BDE, in the points B, E, and C, D; BAC shall be equal to an angle at the center, standing upon half the sum, or half the difference of the arches BC, DE. For, through E, let EF be drawn parallel to AC. The arches CF, DE, intercepted by the parallels AC, EF, are equal^a. Hence the arch BF is the sum or difference of BC, DE, according as A is within or without the circle. But the angle BEF, at the circumference, is equal to an angle at the center upon half the arch BF^b.

There-

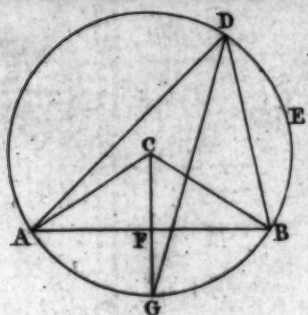


FIG. 55.

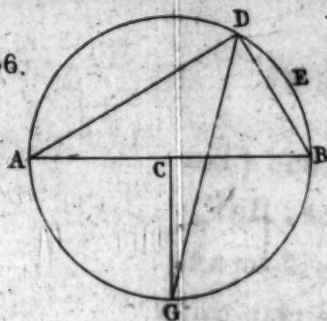


FIG. 56.

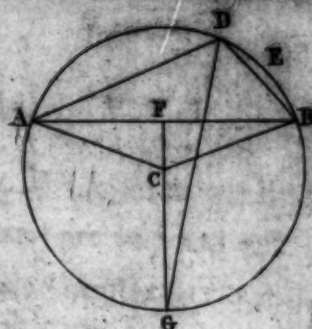


FIG. 57.

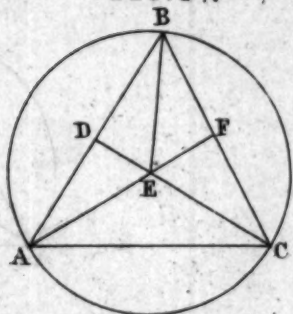


FIG. 58.

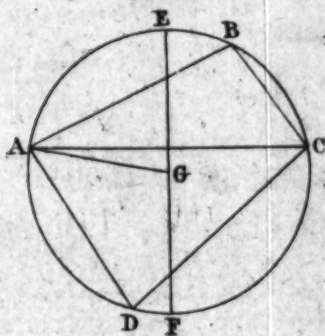


FIG. 59.

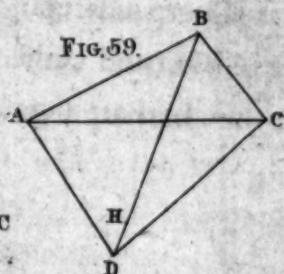


FIG. 60.

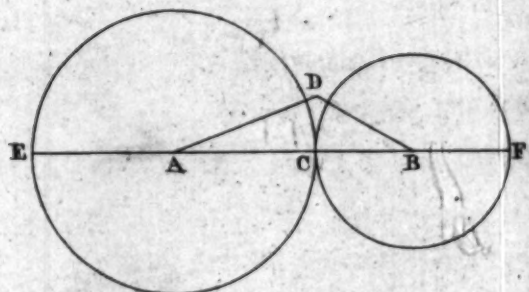
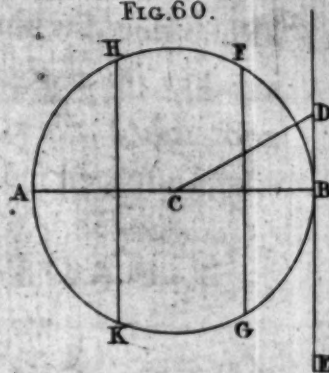


FIG. 61.

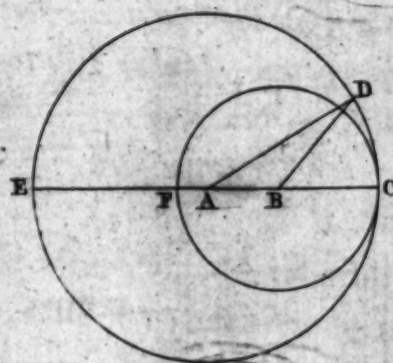


FIG. 63.

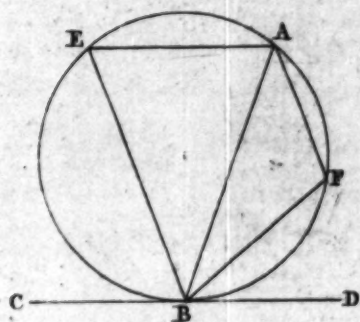


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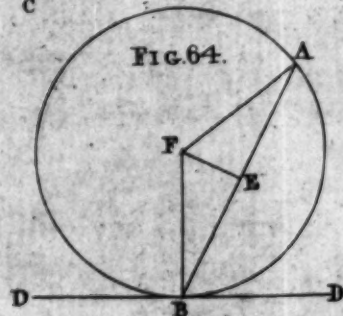


FIG. 62.

1862
The first of the year was a very dry one
and the crops were much injured
by the drought. The wheat was
very poor and the corn was
also much injured. The
cattle and sheep were
also much injured by the
drought. The people were
very poor and the
country was very dry.
The first of the year was a very dry one
and the crops were much injured
by the drought. The wheat was
very poor and the corn was
also much injured. The
cattle and sheep were
also much injured by the
drought. The people were
very poor and the
country was very dry.

Therefore the angle BAC, which is equal to BEF, because of parallel lines, is equal to an angle at the center, upon half the sum, or half the difference, of the intercepted arches BC, DE.

COR. If two chords intersect one another at right angles, within a circle, the opposite arches are together equal to half the circumference.

PROP. XVII. THEOR.

If two chords intersect one another, within or without a circle, the rectangle under the segments of the one will be equal to the rectangle under the segments of the other.

Let AB, CD be two chords intersecting one another in any point E, within or without the circle CBA, the rectangle under AE and EB will be equal to the rectangle under CE and ED. Fig. 66.

Let one of them, as AB, pass through the center F, from which let FG be drawn perpendicular to the other CD, and let FC be joined. In the triangle CFE, the rectangle under the sum and difference of the sides CF, FE, is equal to the rectangle under the sum and difference of the segments of the base CG, GE^a. But the sum of CF, FE is AE, and their difference EB, because AF, FB are each equal to CF; also the sum of CG, GE is CE, and their difference ED, because CD is bisected by FG^b. There-

^a 11. 2.

^b cor. 1.
2. 3.

K

fore

fore the rectangle AEB is equal to the rectangle CED. Consequently the rectangles under the segments of any two chords which intersect in E are equal.

COR. 1. If, from the same point, without a circle, two straight lines be drawn, the one to cut the circle, and the other to touch it, the rectangle under the segments of the former will be equal to the square of the latter.

Fig. 67.

^c 11. 3.

^d cor. 9. 2.

For, if EC be a tangent, CFE is a right angled triangle^c, and the difference of the squares of CF, FE is equal to the square of EC^d.

COR. 2. If, from the same point, without a circle, two straight lines be drawn, the one to cut the circle, and the other to meet the circumference, so that the rectangle under the segments of the former be equal to the square of the latter, the latter will be a tangent.

If EC be the line that meets the circle, and EBA the line that cuts it, so that the square of EC be equal to the rectangle AEB; then EC is a tangent: For, if it met the circumference in any other point, as G, the rectangle GEC would be equal to the rectangle AEB, and therefore equal to the square of EC, which is impossible^c.

^d cor. 2.

6. 2.

Fig. 68.

COR. 3. If, from any point in the circumference of a circle, a perpendicular be let fall upon a diameter, the

the rectangle under the segments of the diameter will be equal to the square of the perpendicular.

COR. 4. Also, if from the same point, a straight line be drawn to one extremity of the diameter, its square will be equal to the rectangle under the whole diameter, and the adjacent segment.

PROP. XVIII. THEOR.

If, through one extremity of a diameter of a circle, a straight line be drawn, to meet the circumference, and any perpendicular to that diameter, the rectangle under its segments, from that point, will be equal to the rectangle under the diameter, and the distance of the same point from the perpendicular.

Let ABE be a circle, of which AB is a diameter; Fig. 69.
let CD be any straight line perpendicular to AB, within or without the circle, and AD any straight line drawn through the point A, to meet the perpendicular CD in D, and the circumference in E; the rectangle DAE will be equal to the rectangle BAC.

For E, B being joined, the angles DCB, DEB are equal, because they are right angles^a; therefore a circle may be described through the four points, D, C, E, B^b; and consequently the rectangle DAE is equal to the rectangle BAC^c.

^a hyp. &
9. 3.

^b 10. 3.
cor. 2. ib.

^c 17. 3.

COR.

COR. If, through any point, in the circumference of a circle, two straight lines be drawn, to meet the circumference, and any perpendicular to the straight line passing through that point and the center, the rectangles under their segments, from the same point, will be equal.

P R O P. XIX. P R O B.

To inscribe a circle in a given triangle.

Fig. 70.

^a 7. 1.

^b 10. 1.

^c cor. 5.
18. 1. &
5. 1.

^d 11. 3.

Let ABC be the given triangle, in which it is required to inscribe a circle. Let the two angles at B and C be bisected ^a by the lines BD, CD meeting each other in the point D, from which let the perpendiculars DE, DF, DG, be drawn to the sides of the triangle ^b. The triangles BDE, BDG having the angles at B equal, the angles at E and G right, and the side BD opposite to the right angles common to both, are equal in every respect ^c. Hence DG is equal to DE. In like manner, DF may be proved equal to DE. Thus, the three perpendiculars are equal. Let then a circle be described about the center D, at the distance of one of them, and it will pass through the three points E, F, G, and be inscribed in the triangle, because the sides of the triangle being each perpendicular to a radius of the circle, touch its circumference ^d.

COR.

COR. 1. Straight lines, drawn from the angular points to the center of the inscribed circle, bisect the angles of the triangle.

COR. 2. In the same manner, may be found the center of a circle that shall touch any three straight lines given in position, provided they be not all parallel, nor all tend to the same point.

PROP. XX. PROB.

To inscribe a square in a given circle.

Let ABCD be the given circle, in which it is required to inscribe a square. Let two diameters AC, BD be drawn to cut each other at right angles, and let their extremities be joined by the lines AB, BC, CD, DA. The quadrilateral formed by these lines is a square: For all its angles are right^a, because each^a 9. 3. of them is an angle in a semicircle; it is a parallelogram^b, because any two adjacent angles are together equal to two right angles; and all its sides are equal, because they are the bases of the triangles AEB, BEC, CED, DEA, which are equal in every respect^c: And^c 4. 1. that square having its angular points in the circumference, is inscribed in the circle.

Fig. 71.

^a 9. 3.

^b 2. cor.

14. 1.

^c 4. 1.

PROP.

PROP. XXI. PROB.

To inscribe a regular pentagon in a given circle.

Fig. 72.

Let EGF be the given circle, in which it is required to inscribe a regular pentagon. Let the radius AB be divided in the point C, so that the rectangle ABC may be equal to the square of AC^a. About the center B, with a radius equal to AC, let a circle be described to intersect EGF in D; and let AD, DC, DB, be joined, and DC produced to meet the circumference in E. The arch BE is a fifth part of the whole circumference. For, let the diameter EAF be drawn; then,

1. The triangles ABD, DBC, are mutually equiangular. For, since BD is equal to AC, the rectangle ABC will be equal to the square of BD; therefore BD is a tangent to the circle that passes through the three points A, C, D^b, and the angle BDC equal to DAC^c, an angle in the alternate segment. Thus, the triangles ABD, DBC have the angles at D and A equal, the angle at B common to both, and therefore the remaining angle BDA equal to the remaining angle BCD^d.

2. In the isosceles triangle ABD, each of the angles at the base is double the angle at the vertex. For, since the angle CBD is equal to BDA, and BDA has been

^a cor. 2.

^b 17. 3.

^c 14. 3.

^d cor. 2.

18. 1.

been proved equal to $\angle BCD$, the angle $\angle CBD$ is equal to $\angle BCD$, and therefore the side CD is equal to BD^c . ^{c cor. 5. 1}
 But BD is equal to CA . Therefore CD is equal to CA , and the angle $\angle CDA$ to $\angle CAD^f$. But the angle $\angle BDC$ was also proved equal to $\angle CAD$. ^{f cor. 4. 1.} Consequently the angle $\angle BDA$ is double the angle $\angle CAD$ or $\angle BAD$.

3. The chord BD is parallel to the diameter EF . For, since the angles $\angle BDC$, $\angle CDA$ are each equal to $\angle CAD$, they are equal to one another. But $\angle CDA$ is equal to $\angle AEC$. Therefore $\angle BDC$ is equal to $\angle AEC$, and they are alternate angles; consequently BD is parallel to EF^g . ^{g 15. 1.}

4. The arches EB , DF are each of them double the arch BD^h , because the angles $\angle EAB$, $\angle DAF$ (being equal to their alternate angles at B and D ,) are each of them double the angle at $\angle BAD$. ^{h cor. 3. 2. 3.}

Hence the arch BD is a fifth part of the semicircumference EDF , and consequently BE a fifth part of the whole circumference.

Now, let the three arches EG , GH , HK , be made each equal to BE , and the remaining arch BK will be equal to the same. Also, let the chords of these five arches BE , EG , GH , HK , KB be drawn, and the polygon which they form will be a regular pentagon inscribed in the circle. For all its sides are equal ^{i 3. 3.}, because they subtend equal arches, and all its angles equal ^{k cor. 3. 7. 3.}, because they stand upon equal parts of the circumference.

COR.

COR. 1. If the circumference of a circle be divided into any greater number of equal parts, the chords of these parts form a regular polygon.

COR. 2. If the radius of a circle be divided, so that the rectangle under the whole, and one of the parts, may be equal to the square of the other, the greater segment is equal to the side of a regular decagon inscribed in that circle.

P R O P. XXII. P R O B.

To inscribe a regular hexagon in a given circle.

Fig. 73.

Let EDB be the given circle, in which it is required to inscribe a regular hexagon. Let A be the center, and B any point in the circumference. About the center B, at the distance BA, let another circle be described, to intersect the former in C, G, and from the points C, B, G, let the three diameters CF, BE, GD, of the given circle, be drawn. Then the chords of the six arches, into which these diameters divide the circumference, will form a regular hexagon. For, since ABC and ABG are equilateral triangles^a, each of the angles GAB, BAC is a third part of two right angles^b. But the three angles GAB, BAC, CAD make two right angles^c. Therefore the angle CAD is also a third part of two right angles,

^a 1. 1.

^b cor. 4.

18. 1.

^c cor. 1.

12. 1.

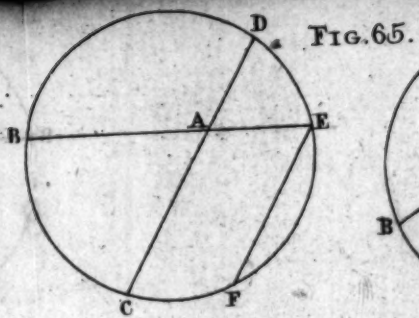
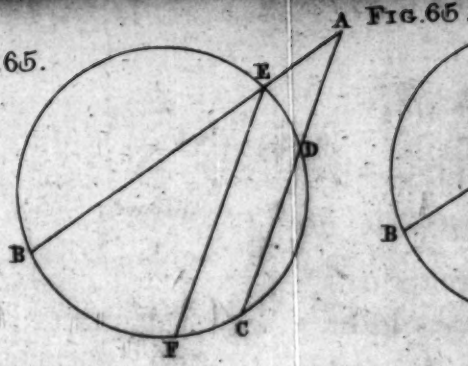


FIG. 65.



A FIG. 66.

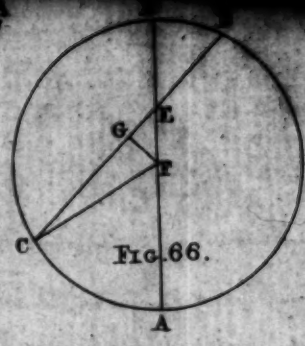
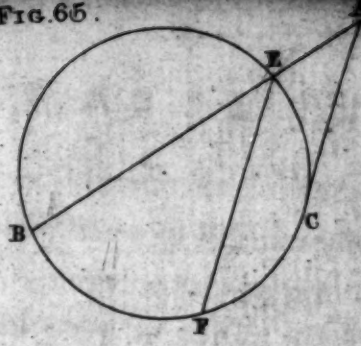


FIG. 68.

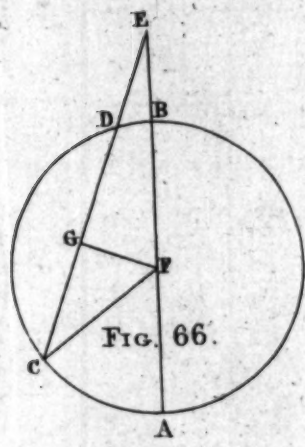


FIG. 69.

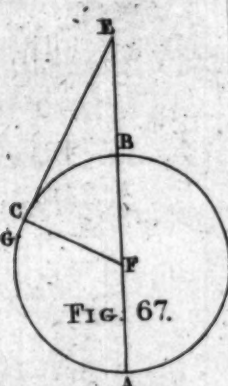


FIG. 70.

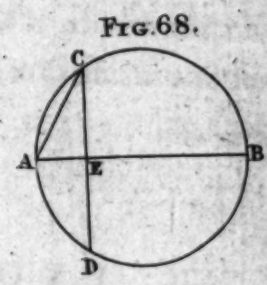


FIG. 71.

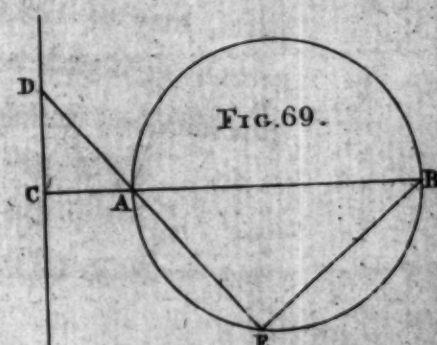


FIG. 72.

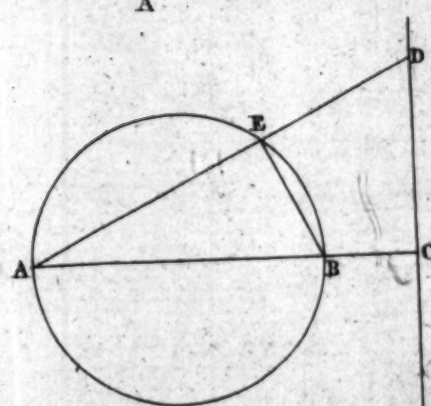


FIG. 73.

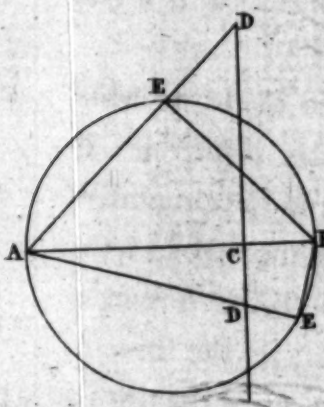


FIG. 74.

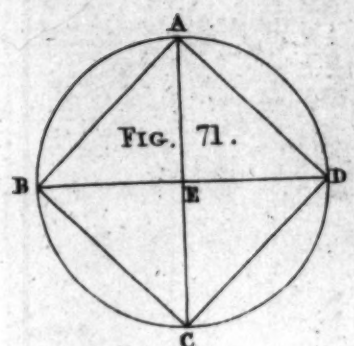
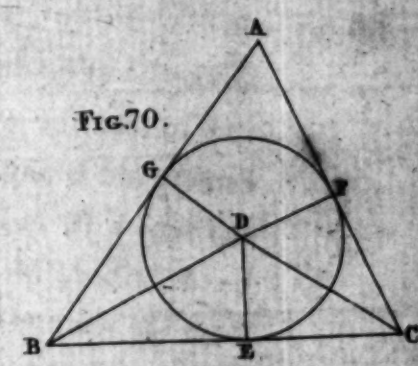


FIG. 76.

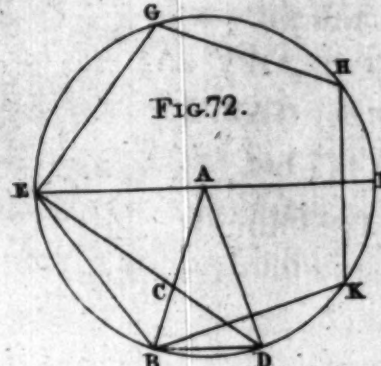


FIG. 77.

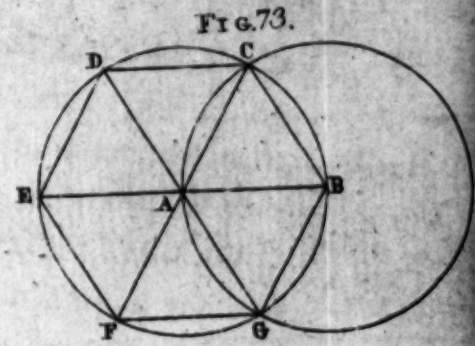


FIG. 78.



angles, and these three angles equal to one another. And, because each of these is equal to its opposite angle ^d, all the six angles at the center A are equal. ^d 12. 1. Consequently the arches upon which they stand are equal ^e; and the figure formed by the chords of these ^e cor. 3. arches, a regular ^f hexagon. ^f cor. 1. 21. 3.

COR. In every circle, a chord equal to the radius subtends a sixth part of the circumference—or, the radius of the circle is equal to the side of the inscribed regular hexagon.

PROP. XXIII. THEOR.

The square of the side of a regular pentagon, inscribed in a circle, is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.

Let BF be the side of a regular pentagon, and BD the side of a regular decagon, inscribed in the circle EDB, of which A is the center. The square of BF shall be equal to the squares of AE, BD. Fig. 74.

Let the radius AB be divided in C, so that the rectangle ABC may be equal to the square of AC ^a, ^a 12. 2. and let FD, DC, CF, FE be joined. Then each of the lines DC, DB, DF, being equal to AC ^b, they ^b 21. 3. are equal to one another. Hence the angle DBC is

L

equal

equal to DCB. But the two angles EFD, DBC being equal to two right angles^e, are equal to the two angles ECD, DCB. Therefore the angle EFD is equal to ECD. From these let the equal^d angles DFC, DCF be taken away, and there remains the angle EFC equal to ECF. Consequently the lines EF, EC are equal^e.

Again, because EC is equal to the sum of AB, AC, it is divided similarly to AB^e, that is, the rectangle ECA is equal to the square of EA: Therefore twice the square of EA is equal to twice the rectangle ECA. Let twice the square of EA be added to both: Then four times the square of EA is equal to twice the rectangle ECA, with twice the square of EA. But four times the square of EA is equal to the square of EB^g, or to the squares of EF, FB^h. Therefore the squares of EF, FB are equal to twice the rectangle ECA, with twice the square of EA. Instead of twice the rectangle ECA, with once the square of EA, let the squares of EC, CAⁱ be taken. Then the squares of EF, FB are equal to the three squares of EC, CA, and EA. From these equals let the equal squares of EF, EC be taken away, and there remains the square of FB, equal to the squares of CA and EA.

Fig. 75. Cor. Hence the side of a regular pentagon may be found by the following construction. Let the diameter EAB be drawn, the radius AD perpendicular to it, and the radius EA bisected in F. About the center

center F, at the distance FD, let a circle be described to intersect EB in C, and let CD be joined. The straight line CD is equal to the side of a regular pentagon inscribed in the circle.

PROP. XXIV. PROB.

About a given circle to describe a regular polygon, of the same number of sides, with a given one inscribed in the same.

Let ABCDEF be the given regular polygon, inscribed in the given circle ACD : It is required to describe about the circle ACD a regular polygon of the same number of sides with ABCDEF. Through the angular points of the given polygon, let tangents to the circle be drawn, intersecting one another in the points G, H, K, L, M, N ; and the figure formed by these tangents is a regular polygon, described about the given circle. Fig. 76.

For, since two tangents, drawn from the same point, are equal ^a, GA is equal to GF ; and thus all the triangles formed upon the sides of the polygon ABCDEF are isosceles. They are also equal to one another. For the angle GAF is equal to an angle in the alternate segment ^b, that is, to an angle at the circumference standing upon the arch AF, and the angle

^a cor. 2.
13. 3.

^b 14. 3.

angle HAB equal to an angle at the circumference, standing upon the arch AB . But the arches AB , AF , subtended by equal lines, are equal^c, and the angles at the circumference, which stand upon them, also equal^d. Therefore the angle GAF is equal to HAB , and consequently the angle GFA equal to HBA . Whence the triangles AGF , AHB , having the sides AB , AF equal, are equal in every respect^e. Thus, the angles at G and H are equal. And, in like manner, may it be proved that all the angles of the polygon $GHKLM$ are equal to one another. Also, the sides GA , AH , are equal. And, in like manner, may it be proved, that all the sides of the polygon are bisected in the points of contact; and, consequently, since their halves are equal, all the sides are equal to one another.

COR. If a regular polygon be described about a circle, the sides of the polygon are bisected in the points of contact.

PROP. XXV. PROB.

To describe a circle about a given regular polygon, and to inscribe another in the same.

Fig. 77.

Let $ABCDEF$ be a given regular polygon, in and about which it is required to describe circles. Let
the

the two angles at A and B be bisected by the lines AG, BG meeting in the point G: Then G is the center of both circles. For, let the perpendiculars GH, GK be drawn from the point G to the sides AB, AF, and let F, G be joined. Then, comparing the triangles AGF, AGB, the sides AF, AB are equal by hypothesis; AG is common to both, and the included angles are equal by construction; therefore the base FG is equal to the base GB, and the angle AFG equal to ABG. But the angle ABG is by construction half the angle ABC. Therefore the angle AFG is half the equal angle AFE, that is, the angle AFE is bisected by the line FG. In like manner, may it be proved, that all the angles of the polygon are bisected by lines drawn from G to the angular points.

Again, because the angles BAF, AFE are equal, their halves GAF, FAG are also equal; therefore the side GF is equal to GA. In the same manner, it may be demonstrated, that all the lines drawn from G to the angular points of the figure are equal to one another. Therefore the circle described about the center G, at the distance GA, will pass through the points A, B, C, D, E, F, and circumscribe the given polygon.

Next, comparing the triangles AGH, AGK, the angles at A are equal, because GA bisects the angle BAF, the angles at H and K are equal, because they are

cor. 5.
18. 1. &
5. 1.

are right angles, and the side GA is common to both. Hence the triangles are equal in every respect: The side GH is equal to GK . In like manner, may it be proved that all the perpendiculars which fall from G , on the sides of the figure, are equal to one another. Therefore the circle described about the center G , at the distance GH , will pass through their extremities: And it will be inscribed in the polygon, because the sides of the polygon, being each of them perpendicular to a radius of the circle, touch its circumference.

COR. Of any regular polygon, the circumscribing and inscribed circles, have the same center.

A P P E N D I X.

PROP. 1. Perpendiculars falling from the extremities of a diameter of a circle, upon any chord in the same, cut off equal segments.

PROP. 2. The square of the side of an equilateral triangle, inscribed in a circle, is triple the square of the radius.

PROP. 3. If two chords in a circle intersect one another at right angles, the sum of the squares of the four segments will be equal to the square of the diameter.

PROP. 4. The base, the perpendicular, and the sum or difference of the sides, being given, to describe the triangle.

PROP.

PROP. 5. If an equilateral triangle be inscribed in a circle, and from the three angular points, straight lines be drawn to the same point in the circumference, one of them shall be equal to the sum of the other two.

PROP. 6. The perpendicular of an equilateral triangle is triple the radius of the inscribed circle.

PROP. 7. A regular octagon, inscribed in a circle, is equal to the rectangle under the sides of the inscribed and circumscribing squares.

PROP. 8. The vertical angle, the difference of the segments of the base, and the sum or difference of the sides are given, to construct the triangle.

PROP. 9. If, from the extremities of any chord in a circle, straight lines be drawn, to any point in the diameter to which it is parallel, the sum of their squares will be equal to the sum of the squares of the segments of the diameter.

PROP. 10. If, from two points, in the diameter of a circle, equally distant from the center, two straight lines be drawn to meet each other in the circumference, the sum of their squares will be equal to the sum of the squares of the segments into which the diameter is divided in one of these points.

PROP. 11. If, from a point without a circle, two tangents be drawn, and from one of the points of contact, a perpendicular be let fall upon the diameter passing through the other, it will be bisected by a straight

straight line joining the point without the circle, and the farthest extremity of the diameter.

PROP. 12. The difference of the angles at the base, the difference of the segments of the base, and the sum or difference of the sides being given, to construct the triangle.

PROP. 13. If, from the extremities of the base of any triangle, perpendiculars be let fall upon the opposite sides, a straight line drawn through their intersection from the vertex, will be perpendicular to the base.

PROP. 14. From a point, without a circle, let two tangents be drawn, and from the points of contact two chords, to the same point in the circumference; then, if a secant be drawn from the point without the circle, parallel to the one chord, and meeting the other, a perpendicular to the parallel chord, from the point of concurrence, will bisect it.

PROP. 15. If, from any point in the circumference of a circle, two chords be drawn, and the intercepted arch be bisected by a perpendicular falling upon either chord, that perpendicular will add to the less, or cut off from the greater chord, a part equal to half their difference.

PROP. 16. The difference of the angles at the base, the difference of the segments of the base, and the perpendicular being given, to construct the triangle.

PROP.

PROP. 17. If, from a point without a circle, two straight lines be drawn, to touch or cut the circumference, and from the same point a secant be drawn, through the line joining the points of contact, or the intersection of the lines joining the opposite points of section, it will be so divided, that the rectangle under the whole line, and its middle segment, shall be equal to the rectangle under the extreme segments.

PROP. 18. If the opposite sides of a quadrilateral, inscribed in a circle, be produced to meet, the square of the line joining the points of concurrence will be equal to the sum of the squares of two tangents drawn from these points.

PROP. 19. If, from a point without a semicircle, two tangents be drawn, and the points of contact be joined to the opposite extremities of the diameter, the straight line which passes through their intersection, and the point without the semicircle, will be perpendicular to the diameter.

PROP. 20. The vertical angle, the perpendicular, and the perimeter, being given, to construct the triangle.

28

E L E M E N T S
O F
G E O M E T R Y.

B O O K IV.

D E F I N I T I O N S.

1. **W**HEN one magnitude contains another a certain number of times, without a remainder, the former is said to be a multiple of the latter, and the latter a part of the former.

2. When several magnitudes are multiples of as many others, and each contains its part the same number of times, the former are said to be equimultiples of the latter, and the latter like parts of the former.

3. Betwixt any two finite magnitudes of the same kind, there subsists a certain relation, in respect to
quantity,

quantity, which is called their Ratio. The two magnitudes compared are called the Terms of the Ratio; the first, the Antecedent; and the last, the Consequent.

4. Of four magnitudes, the ratio of the first to the second is said to be the same with (or equal to) the ratio of the third to the fourth, when the first contains any part whatever of the second, just as oft as the third contains the like part of the fourth.

5. But the ratio of the first to the second is said to be greater than the ratio of the third to the fourth, when the first contains a part of the second oftener than the third contains a like part of the fourth, and less when not so often.

6. When two ratios are equal, their terms are said to be proportional.

7. The equality of ratios, or proportionality of magnitudes, is expressed thus, $A : B = C : D$, (the ratio of A to B is equal to the ratio of C to D), or $A : B :: C : D$, (A is to B as C to D). And the whole expression is named an Analogy or Proportion.

8. The homologous terms of an analogy, are the Antecedents or the Consequents of the Ratios.

9. Magnitudes are said to be in continued proportion, when the ratios of the first to the second, of the second to the third, of the third to the fourth, &c. are all equal.

10. Of magnitudes in continued proportion, the ratio of the first to the third is called the duplicate ratio of the first to the second, the ratio of the first to the fourth, the triplicate of the first to the second, &c.; also, the ratio of the first to the second is called the subduplicate of the first to the third, the subtriplicate of the first to the fourth, &c.

11. Of any number of magnitudes of the same kind, the ratio of the first to the last is said to be compounded of the ratios of the first to the second, the second to the third, the third to the fourth, and so on to the last.

12. Of any analogy, the magnitudes are said to be proportional by *Inversion*, when the second is to the first as the fourth to the third.

13. By *Alternation*, when the first is to the third as the second to the fourth.

14. By *Composition*, when the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth.

15. By *Division*, when the difference of the first and second is to the second, as the difference of the third and fourth is to the fourth.

16. By *Conversion*, when the first is to the sum of the first and second, as the third to the sum of the third and fourth. Also, when the first is to the difference of the first and second, as the third to the difference of the third and fourth.

17. By *Mixing*, when the sum of the first and second is to their difference as the sum of the third and fourth to their difference.

18. If there be any number of magnitudes, and as many others, which, taken two and two, are continually proportional, either in a direct order, (that is, if the first be to the second of the former, as the first to the second of the latter, the second to the third of the former, as the second to the third of the latter, &c.) or in a cross order, (that is, if the first be to the second of the former, as the last but one to the last of the latter, the second to the third of the former, as the last but two to the last but one of the latter, &c.), they are said to be proportional by *Equality*, when the first is to the last of the former, as the first to the last of the latter.

A X I O M S.

1. Equal magnitudes contain the same or equal magnitudes, the same number of times.

2. Of two magnitudes, that which contains a third magnitude the greater number of times is the greater.

3. Equimultiples, or like parts of the same, or equal magnitudes, are equal.

4. A multiple, or part of a greater magnitude, is greater than the same multiple, or part of a less.

5. All ratios of equality are the same.
6. Ratios that are equal to the same, or to equal ratios, are equal to one another.
7. Two equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to each of two equal magnitudes.
8. If one ratio be greater than another, any ratio equal to the former will be greater than the latter; also the former will be greater than any ratio equal to the latter.

L E M M A I.

If there be any number of magnitudes equimultiples of as many, the sum of the former will be the same multiple of the sum of the latter, that any one of them is of its part.

Fig. 78. Let any number of magnitudes AB, CD be equimultiples of as many E, F ; the sum of AB, CD will be the same multiple of the sum of E, F , that AB is of E . For, since AB is a multiple of E , it consists of a certain number of parts, each equal to E^a , let these be Am, mn, nB . In like manner, CD consists of a certain number of parts, each equal to F , let these be Cp, pq, qD . And, because AB, CD , are equimultiples of E, F , the number of parts in each is the

^a def. 1. 4.

the same ^b. But the sum of Am , Cp is equal to the ^b def. 2. 4 sum of E , F ; the sum of mn , pq is equal to the sum of E , F ; and the sum of nB , qD equal to the sum of E , F . Therefore the sum of AB , CD contains the sum of E , F the same number of times, without a remainder, that one of them AB contains its part E .

COR. If AB contains E just as oft as CD contains F , the sum of AB , CD will also contain the sum of E , F the same number of times, although AB , CD be not equimultiples of E , F .

LEMMA II.

If, of two magnitudes, there be any number of equimultiples, the sum of the multiples of the one, and the sum of the multiples of the other, will also be equimultiples of the same magnitudes.

Of the two magnitudes A , B , let there be any number of equimultiples, as CD , EF ; DG , FH , the sum of CD , DG , and the sum of EF , FH will also be equimultiples of A , B . For, since CD , EF are equimultiples of A , B , there are as many magnitudes equal to A in CD , as there are magnitudes equal to B in EF ^a. In like manner, there are as many magnitudes equal to A in DG as there are

are

Fig. 79

are magnitudes equal to B in FH. Consequently the whole CG contains A the same number of times that EH contains B, and there is no remainder; therefore CG, EH are equimultiples of A, B.

COR. If two magnitudes be equimultiples of other two, any equimultiples of the former will also be equimultiples of the latter. And any like parts of the latter will be like parts of the former.

L E M M A III.

If, from any magnitude, its half be taken away, and from the remainder its half, and so on continually; there shall at length remain a part of that magnitude less than any magnitude proposed.

Fig. 30.

From AB let its half BC be taken away, and from the remainder AC let its half CD be taken away, and so on continually, there shall at length remain a part of AB less than any magnitude proposed, as FG. For, let magnitudes GH, HK, KL, each equal to FG, be continually added to FG, till the whole LF shall be a multiple of FG, greater than AB*. And, let the number of divisions BC, CD, DE, EA, in AB, be the same with the number of parts in LF. Then, because AB is less than LF, if BC, which is the half of AB, and LK, which is not greater than the half

* post. 4.
I.

half of LF, be taken from them, the remainder AC will be less than the remainder FK. In like manner, AD is less than FH, and AE, (which is evidently a part of AB), less than FG.

COR. If, from any magnitude, more than its half be taken away, and from the remainder more than its half, and so on continually, there shall at length be a remainder less than any magnitude proposed.

PROPOSITION I.

Magnitudes have the same ratio as their equimultiples.

Let CG, DL be equimultiples of A, B ; $A : B ::$ Fig. 61. $CG : DL$. For, let Bn be any part of B, less than A ; CE, EF, FG, each equal to A ; DH, HK, KL, each equal to B ; and Dn, Hn, Kn, each equal to Bn. Then, as often as A contains Bn, so often does CE contain Dn ; EF, Hn ; FG, Kn^a ; and so often^a ax. 1. 4. does CG contain the sum of Dn, Hn, Kn^b. But the^b cor. lem. 1. sum of Dn, Hn, Kn, is the same part of DL that Dn is of DH^c, or Bn of B. Therefore $A : B :: CG :$ ^c lem. 1. DL ^d def. 4. 4.

COR. Magnitudes have the same ratio as their like parts.

N P R O P.

PROPOSITION II.

If, instead of two homologous terms of an analogy, any equimultiples, or like parts of them, be taken, the magnitudes will still be proportional.

Fig. 82.

Let $A : B :: C : D$. And, 1st, instead of B, D , let B_m, D_m , like parts of them, be taken, then $A : B_m :: C : D_m$.

^a cor. lem.
2.

^b def. 4. 4.

For, let B_n, D_n be any like parts of B_m, D_m ; they will also be like parts of B, D ^a; therefore A contains B_n as often as C contains D_n ^b. Whence $A : B_m :: C : D_m$ ^b.

Fig. 83.

2. Instead of A, C , let A_m, C_m , like parts of them, be taken, then $A_m : B :: C_m : D$.

^c def. 5. 4.

For, if one of these ratios, as $A_m : B$, be greater than the other $C_m : D$, A_m contains some part of B , suppose B_n , oftener than C_m contains D_n , the like part of D ^c. Therefore also $mp = A_m$ contains B_n oftener than $mq = C_m$ contains D_n ; and so on. Consequently the whole of A contains B_n oftener than the whole of C contains D_n ; which is contrary to the hypothesis.

Fig. 84.

3. Instead of B, D , let BM, DM , equimultiples of them, be taken; then $A : BM :: C : DM$.

For, let AE, CF , be the same multiples of A, C , that BM, DM are of B, D . Then $AE : BM (:: A :$

B

$B^f :: C : D) :: CF : DM$ *r.* But A, C are like^t *r.* 4. parts of AE, CF; therefore $A : BM :: C : DM$.

4. Instead of A, C, let AM, CM, equimultiples of them, be taken, $AM : B :: CM : D$.

For, let BE, DF be the same multiples of B, D, Fig. 85. that AM, CM are of A, C. Then $AM : BE (:: A : B^f :: C : D) :: CM : DF^f$. But B, D are like parts of BE, DF; therefore $AM : B :: CM : D$.

PROPOSITION III.

If four magnitudes be proportional, according as the first is greater, equal, or less, than the second, the third will be greater, equal, or less, than the fourth.

Let $A : B :: C : D$. And, 1st, Let A be greater Fig. 86. than B, then C will be greater than D.

For, let AE be equal to B; Bn, a part of B, less than EG, the difference of A, B^a; and Dn the like^a lem. 3. part of D. Then it is evident that A contains Bn oftener than (AE, that is,) B contains Bn. But C contains Dn just as often as A contains Bn^b, and D^b def. 4. 4. contains Dn just as often as B contains Bn^c. There-^c def. 2. 4. fore C contains Dn oftener than D contains Dn; and consequently C is greater than D^d. ^d ax. 2. 4.

2. Let A be less than B, then C will be less than Fig. 87. D.

For,

For, let AE be equal to B , Bn , a part of B , less
^a lem. 3. than either A or EG , and Dn the like part of D .
 Then it is evident that (AE , that is) B contains Bn
 oftener than A contains Bn . But D contains Dn just
^c def. 2. 4. as often as B contains Bn , and C contains Dn just
^b def. 4. 4. as often as A contains Bn . Therefore D contains
 Dn oftener than C contains Dn ; and consequently D
^d ax. 2. 4. is greater than C , or C less than D .

3. Let A be equal to B , then C will be equal
 to D .

For, from what has been demonstrated, C can
 neither be greater nor less than D .

COR. 1. The greater of two unequal magnitudes
 has a greater ratio to the same magnitude than the
 less has. And conversely, of two magnitudes, that
 which has the greater ratio to any third magnitude is
 the greater. If A be greater than B , $A : C$ is greater
 than $B : C$; for A contains a part of C less than
 $A - B$, oftener than B contains the same. If $A : C$
 be greater than $B : C$, A is greater than B , because
 A contains some part of C oftener than B does.

COR. 2. Those magnitudes which have the same
 ratio to the same magnitude are equal. If $A : C =$
 $B : C$, A is equal to B , because otherwise the ratios
 would not be equal.

PROPO.

PROPOSITION IV.

If four magnitudes be proportional, they are also proportional by inversion.

Let $A : B :: C : D$, then $B : A :: D : C$. Fig. 88.

For, let A_m , C_m be like parts of A , C . Then $A_m : B :: C_m : D$, and any multiple of A_m is to B ^{a 2. 4} as the same multiple of C_m is to D . Therefore, according as B is greater, equal, or less, than any multiple of A_m , D is greater, equal, or less, than the same multiple of C_m . Consequently B contains A_m just ^{b 3. 4} as oft as D contains C_m , and $B : A :: D : C$.

COR. 1. Those magnitudes, to which the same magnitude has the same ratio, are equal. If $C : A :: C : B$, A is equal to B . For, by inversion, $A : C :: B : C$; and therefore $A = B$. c 2. c. 3. 4.

COR. 2. The same magnitude has a greater ratio to the less of two unequal magnitudes than it has to the greater. And conversely, Of two magnitudes, that to which any third magnitude has the greater ratio is the less. If A be greater than B , $C : B$ is greater than $C : A$. For these ratios cannot be equal, because from thence it would follow that A , B are also equal^d, which is contrary to hypothesis. Neither ^{d 1. c. 4. 4} can $C : A$ be greater than $C : B$; because then C would contain some part of A oftener than the like part

part of B; that is, it would contain a greater magnitude oftener than a less; which is absurd. If $C : B$ be greater than $C : A$, A is greater than B. For, since C contains some part of B oftener than it contains the like part of A, that part of B must be less than the like part of A, and consequently the whole of B less than the whole of A.

ax. 4. 4.

PROPOSITION V.

If the first be the same multiple, or part, of the second, that the third is of the fourth, the four magnitudes shall be proportional. And conversely, Of four proportional magnitudes, if the first be a multiple, or part, of the second, the third shall be the same multiple, or part, of the fourth.

Fig. 89.

Let A be the same multiple of B that C is of D, then $A : B :: C : D$.

For, let B_n , D_n be like parts of B, D. Then, because A, C, are equimultiples of B, D, and B, D equimultiples of B_n , D_n ; A, C will also be equimultiples of B_n , D_n ; that is, A contains any part whatever of B as often as C contains the like part of D. Therefore $A : B :: C : D$.

cor. lem.
2.

Fig. 90.

Next, let A be the same part of B that C is of D; then B is the same multiple of A that D is of C; there-

therefore $B : A :: D : C$; and, by inverſion, $A : B :: C : D$ ^b.

Converſely, If $A : B :: C : D$, and A a multiple or part of B , C is the ſame multiple or part of D . For, let E be the ſame multiple or part of D that A is of B . Then, $A : B :: E : D$; but $A : B :: C : D$; therefore $C : D :: E : D$. Hence C is equal to E ^c, ^{2. c. 3. 4.} and the ſame multiple or part of D that A is of B .

COR. If four magnitudes be proportional, and the firſt a multiple of any part of the ſecond, the third ſhall be the ſame multiple of the like part of the fourth. And converſely^d. ^{2. 4.}

PROPOSITION VI.

If four magnitudes be proportional, according as the firſt is greater, equal, or leſs than the third, the ſecond will be greater, equal, or leſs than the fourth.

Let $A : B :: C : D$, then, if A be greater than C , Fig. 91. B will be greater than D .

For, ſince A is greater than C , $A : B$ is greater than $C : B$ ^a; but $A : B = C : D$; therefore $C : D$ is ^a 1. c. 3. 4. greater than $C : B$ ^b, and conſequently B greater than ^b ax. 8. D ^c. In like manner, if A be equal to C , it may be ^c 2. c. 4. 4. proved that B is equal to D ; and, if A be leſs than C , that B is leſs than D .

P R O.

PROPOSITION VII.

If four magnitudes of the same kind be proportional, they are also proportional by alternation.

Fig. 91. Let $A : B :: C : D$, then $A : C :: B : D$.

For, let Cn, Dn be like parts of C, D . Then
 * c. 1. 4. $A : B (:: C : D) :: Cn : Dn^a$, and $A : B ::$ any
 ax. 6. multiple of Cn : the same multiple of Dn^b . There-
 * 1. 4. fore, according as A is greater, equal, or less, than
 any multiple of Cn ; B is greater, equal, or less than
 ‡ 6. 4. the same multiple of Dn^c . Consequently A con-
 tains Cn just as often as B contains Dn — and $A : C$
 $:: B : D$.

PROPOSITION VIII.

If four magnitudes be proportional, they are also proportional by composition.

Fig. 92. Let $A : B :: C : D$, then $A + B : B :: C + D : D$.

For, let Bn, Dn be like parts of B, D ; then, by hypothesis, A contains Bn just as often as C contains Dn . If, therefore, a certain number of times Bn be added to A , and the same number of times Dn to C , the former sum will contain Bn as often as the latter
 con-

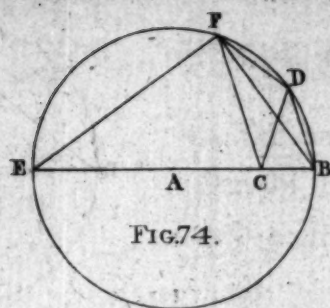


FIG. 74.

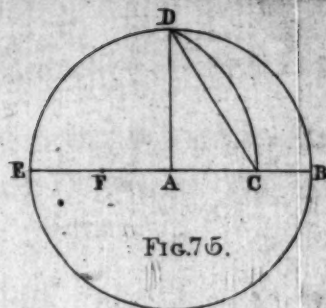


FIG. 75.

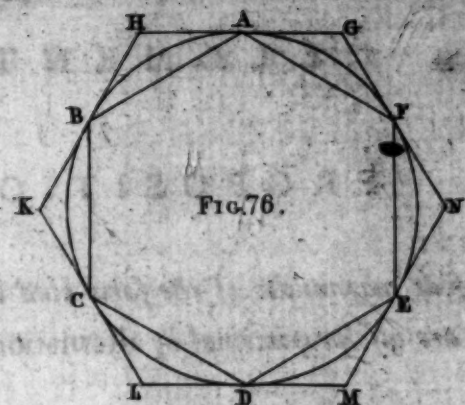


FIG. 76.



FIG. 78.

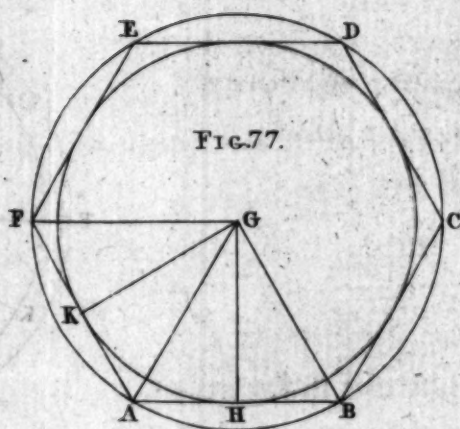


FIG. 77.

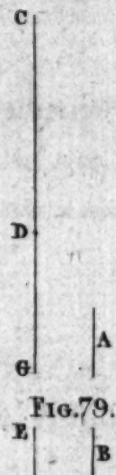


FIG. 79.

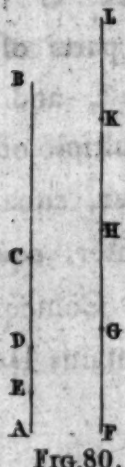


FIG. 80.

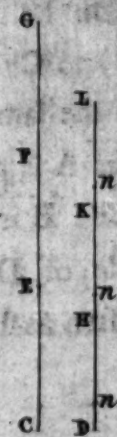


FIG. 81.



FIG. 82.

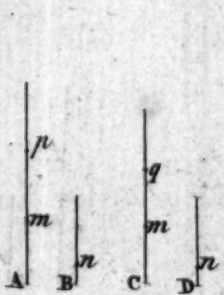


FIG. 83.



FIG. 84.

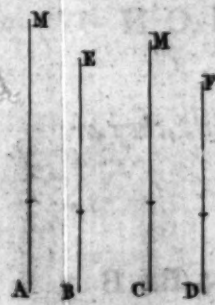


FIG. 85.



FIG. 86.



FIG. 87.



FIG. 88.



FIG. 89.

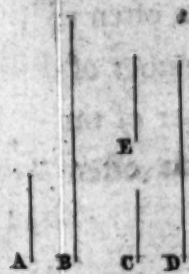


FIG. 90.



FIG. 91.



FIG. 92.

18

THE HISTORY OF THE

REIGN OF

CHARLES THE FIRST

BY

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OF THE UNIVERSITY OF OXFORD

IN TWO VOLUMES

VOLUME THE FIRST

THE FIRST PART

OF THE REIGN

OF CHARLES THE FIRST

FROM HIS BIRTH

TO HIS DEATH

IN THE YEAR 1649

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contains Dn . Thus, the sum of A and B contains Bn as often as the sum of C and D contains Dn , or $A + B : B :: C + D : D$.

PROPOSITION IX.

If any number of magnitudes be continually proportional to a like number of magnitudes, taken two and two, either in a direct or cross order, they are proportional by equality.

Let there be three magnitudes A, B, C , and other three D, E, F , which, taken two and two, are proportional in a direct order, viz. $A : B :: D : E$, and $B : C :: E : F$, then $A : C :: D : F$. Fig. 93.

For, let Cn, Fn be like parts of C, F , and CP, FQ any equimultiples of Cn, Fn ; and suppose A greater than CP . Then, because $B : C :: E : F$, by inversion, $C : B :: E : F^a$, taking like parts of homologous terms $Cn : B :: Fn : E^b$, and taking equimultiples of these $CP : B :: FQ : E^b$. But, since A is greater than CP , $A : B$ is greater than $CP : B^c$; therefore also $A : B$ or $D : E$ is greater than $FQ : E^d$, and consequently D greater than FQ^e . In like manner, if $A = CP$, $D = FQ$, and, if less, less. Thus, according as A is greater, equal, or less, than any multiple of Cn , D is greater, equal, or less, than the

O

same

same multiple of F_n . Wherefore A contains C_n as often as D contains F_n , and $A : C :: D : F$.

Fig. 94.

Next, let there be three magnitudes A, B, C , and other three, D, E, F , which, taken two and two, are proportional in a cross order, viz. $A : B :: E : F$, and $B : C :: D : E$, then $A : C :: D : F$.

For, let B_n, C_n, F_n , be any like parts of B, C, F , and BO, CP, FQ , any equimultiples of B_n, C_n, F_n , respectively. Then, because $A : B :: E : F$, $A : B_n :: E : F_n^e$, and $A : BO :: E : FQ^e$. Again, because $B : C :: D : E$, $C : B :: E : D^f$, $C_n : B_n :: E : D^g$, and $CP : BO :: E : D^h$. But, since A is greater than CP , $A : BO$ is greater than $CP : BO^i$; therefore also $E : FQ$ greater than $E : D^k$, and consequently D greater than FQ , as before.

Now, let there be any number of magnitudes, A, B, C, D, E and a like number F, G, H, K, L which, taken two and two, are continually proportional in a direct order, $A : B :: F : G$, $B : C :: G : H$, $C : D :: H : K$, &c.; then, by equality, $A : E :: F : L$. For, taking them three and three, A, B, C , and F, G, H , $A : C :: F : H$; A, C, D , and F, H, K , $A : D :: F : K$, &c.

Also, let there be any number of magnitudes A, B, C, D, E and a like number F, G, H, K, L which, taken two and two, are continually proportional

tional in a cross order, $A : B :: K : L$, $B : C :: H : K$, $C : D :: G : H$, &c. then, by equality, $A : E :: F : L$. For, taking them three and three, A , B , C , and H , K , L ; $A : C :: H : L$; A , C , D , and G , H , L , $A : D :: G : L$, &c.

COR. 1. If the consequent terms in one analogy be antecedents in another, the remaining terms of the former will be the antecedents, and the remaining terms of the latter the consequents of a third analogy.

COR. 2. If either the extreme or mean terms of one analogy be the extremes of another, the remaining terms of the former will be the extremes, and the remaining terms of the latter the means of a third analogy.

COR. 3. Two ratios are equal, if each be compounded of the same number of ratios that are equal, each to each, either in a direct or cross order.

PROPOSITION X.

If four magnitudes be proportional, they are also proportional by division.

Let $A : B :: C : D$, then $A - B : B :: C - D : D$. Fig. 92.

1. When A is greater than B . Let Bn , Dn , be like parts of B , D . Then, since A contains Bn as often

^a def. 4. 4. often as C contains Dn^a , if any number of times Bn be taken from A, and as many times Dn from C, the former remainder will contain Bn as often as the latter contains Dn . Thus, $A - B$ contains Bn as often as $C - D$ contains Dn : Therefore $A - B : B :: C - D : D$.

2. When A is less than B. By inversion $B : A ::$
^b 4. 4. $D : C^b$, and, from what has been demonstrated,

$$B - A : A :: D - C : C$$

$$\text{But } A : B :: C : D$$

^c 9. 4. Therefore, by equality $B - A : B :: D - C : D^c$.

PROPOSITION XI.

If four magnitudes be proportional, they are also proportional by conversion.

Let $A : B :: C : D$, then $A : A + B :: C : C + D$, and $A : A - B :: C : C - D$.

^a 4. 4. For, by inversion, $B : A :: D : C^a$.

^b 8. 4. By composition, $A + B : A :: C + D : C^b$, or, by division, $A - B : A :: C - D : C^c$. And by inversion $A : A + B :: C : C + D^a$, and $A : A - B :: C : C - D^a$.

PROPO.

PROPOSITION XII.

If four magnitudes be proportional, they are also proportional by mixing.

Let $A : B :: C : D$, then $A + B : A - B :: C + D : C - D$.

For, by composition^a, $A + B : B :: C + D : D$. ^a 8. 4.

Also, by division^b and inversion^c, $B : A - B :: D : C - D$. ^b 10. 4. ^c 4. 4.

Therefore by equality^d, $A + B : A - B :: C + D : C - D$. ^d 9. 4.

PROPOSITION XIII.

If there be any number of equal ratios, whose terms are all of the same kind, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $A : B = C : D = E : F$, then $A : B :: A + C + E : B + D + F$. Fig. 95.

For, let Bn, Dn, Fn , be like parts of B, D, F , and the sum of Bn, Dn, Fn , will be the same part of the sum of B, D, F , that Bn is of B ^a. But, since C contains Dn , and E, Fn , just as often as A contains Bn ; ^a Lem. 1.
it

it is evident that the sum of A, C, E, will contain the
^a cor. lem. sum of Bn, Dn, Fn, the very same number of times ^b.
 1. Therefore $A : B :: A + C + E : B + D + F$.

PROPOSITION XIV.

If there be two equal ratios, whose terms are of the same kind, as one antecedent is to its consequent, so is the difference of the antecedents to the difference of the consequents.

Let $A : B :: C : D$, then $A : B :: A - C : B - D$.

^a 7. 4. For, by alternation ^a, $A : C :: B : D$, by conver-
^b 11. 4. sion ^b $A : A - C :: B : B - D$, and by alternation ^a
 $A : B :: A - C : B - D$.

COR. From this, and the preceding proposition, it follows, that a ratio is not changed by either augmenting or diminishing each of its terms by the homologous terms of an equal ratio.

PROPOSITION XV.

If the consequent terms in two analogies be the same, the sum of the first antecedents will be to their consequent as the sum of the last antecedents to their consequent.

Let

Let $A : B :: C : D$, and $E : B :: F : D$, then $A + E : B :: C + F : D$.

For, since $A : B :: C : D$

And (by inversion ^a) $B : E :: D : F$ ^a 4. 4.

By equality ^b $A : E :: C : F$. ^b 9. 4.

By composition ^c $A + E : E :: C + F : F$ ^c 8. 4.

But $E : B :: F : D$

Therefore, by equality ^b $A + E : B :: C + F : D$.

COR. 1. The difference of the first antecedents is to their consequent as the difference of the last antecedents to their consequent.

COR. 2. If the consequent terms, in any number of analogies, be the same, the sum of the first antecedents is to their consequent as the sum of the last antecedents to their consequent.

COR. 3. If the antecedent terms be the same, the first antecedent is to the sum or difference of its consequents as the last antecedent to the sum or difference of its consequents.

PROPOSITION XVI.

If four magnitudes of the same kind be proportional, the sum of the greatest and least is greater than the sum of the other two.

Let

Fig. 96. Let $AE : BG :: C : D$, and AE the greatest term;
 then D is the least^a, and $A + D$ greater than $B + C$.
 For, let $AF = C$ and $BH = D$, then $AE : BG ::$
^a 3. 4. & $EF : GH$ ^b. Hence EF is greater than GH ^a. There-
^b 6. 4. fore AE is greater than GH added to C , and the sum
 of AE and D greater than the sum of BG and C .

COR. If three magnitudes be proportional, the middle term is less than half the sum of the extremes.

PROPOSITION XVII.

If there be any number of ratios, and as many others, equal to them, each to each, the ratio that is compounded of ratios that are the same with the former, each with each, taken in direct order, is equal to the ratio that is compounded of ratios, which are the same with the latter, each with each, taken in any order whatever.

Let there be any number
 of ratios $A : B, C : D, E : F, G : H$
 And as many others $K : L, M : N, O : P, Q : R$
 equal to them, each to each, in direct order.

Let the ratio of m to r be com-
 pounded of the ratios $m : n, n : p, p : q$
 $q : r$ equal to the first ratios, each to each, in direct
 order. And let the ratio of a to e be compounded

of

of $a:b, b:c, c:d, d:e$, equal to the second ratios, each to each, in any order whatever. Then shall the ratio of m to r be equal to the ratio of a to e .

1. Let there be two ratios, $A:B, C:D; m, n, p$ and let $a:b = K:L, b:c = K:L, M:N; a, b, c$ $M:N$. Then, because $m:n (= A:B = K:L) = a:b$, and $n:p (= C:D = M:N) = b:c$, by equality * 9. 4. $m:p = a:c$.

Again, let $a:b = M:N, b:c = K:L$. Then, because $m:n (= A:B = K:L) = b:c$; and $n:p (= C:D = M:N) = a:b$, by equality * $m:p = a:c$. Hence the ratio that is compounded of $K:L, M:N$ is the same with that which is compounded of $M:N, K:L$.

2. Let there be three $A:B, C:D, E:F, m, n, p, q$ ratios; then, if $a:b = K:L, M:N, O:P, a, b, c, d$ $K:L, b:c = M:N, c:d = O:P$; or, if $a:b = O:P, b:c = M:N, c:d = K:L$; it follows, in like manner, by equality *, that $m:q = a:d$.

Hence the ratio that is compounded of $K:L, M:N, O:P$, is the same with that which is compounded of $O:P, M:N, K:L$.

Also, the ratio that is compounded of the first or last of these, and of the compound ratio of the other

P

two,

* That is, of ratios that are the same with these, according to the definition of compound ratio.

two, is the same with that which is compounded of all the three.

Again, let them be taken in any other order, as $a : b = M : N$, $b : c = O : P$, $c : d = K : L$. Then, $m : n = c : d$, $n : p = a : b$, $p : q = b : c$; therefore the ratio of m to q is the same with that which is compounded of $c : d$, $a : b$, $b : c$; or of $c : d$, $a : c$; or of $a : c$, $c : d$; that is, $m : q = a : d$.

3. Let there be $A : B, C : D, E : F, G : H, m : n, p : q, r$ four ratios; and $K : L, M : N, O : P, Q : R, a : b, c : d, e$ let the second ratios be taken in any order whatever, as $a : b = M : N$, $b : c = Q : R$, $c : d = K : L$, $d : e = O : P$. Then, because $m : n = c : d$, $n : p = a : b$, $p : q = d : e$, the ratio of m to q is the same with that which is compounded of $a : b$, $c : d$, $d : e$; or of $a : b$, $c : e$. But $q : r = b : c$. Therefore the ratio of m to r is the same with that which is compounded of $a : b$, $c : e$, $b : c$; or of $a : b$, $b : c$, $c : e$, which is the ratio of a to e .

The same reasoning will hold for any greater number of ratios.

COR. Instead of any one of the ratios, of which any ratio is compounded, an equal ratio may be substituted; instead of two or more, the ratio compounded of them may be substituted; also, the order of the ratios may be changed in any manner whatever; and the compound ratio will still be the same.

PROPO.

PROPOSITION XVIII.

If, from two unequal magnitudes, two magnitudes, having a given ratio, may be taken away, so that either of the remainders shall be less than any magnitude proposed, the two wholes shall have to each other the same ratio as the magnitudes taken from them.

Let AB, CD be the two unequal magnitudes, of which AB is the greater, and $M : N$ the given ratio. It is supposed that, from AB, CD, there may be taken two magnitudes in the ratio of M to N, so that the remainder of AB shall be less than any magnitude proposed, and likewise that from AB, CD there may be taken two magnitudes in the ratio of M to N, so that the remainder of CD shall be less than any magnitude proposed. It is to be proved that $AB : CD :: M : N$. Fig. 97.

1. The ratio of AB to any magnitude CO, less than CD, is greater than the ratio of M to N. For, since there may be taken two magnitudes from AB, CD, in the ratio of M to N, so that the remainder of CD shall be less than DO, let these be AE, CF; CF will therefore be greater than CO. Let $AR : CO :: AE : CF$, then AR will be less than AE^a, and consequently less than AB. Hence the ratio AB : CQ is greater than AR C : O^b, or M : N.

^a 4. 4. 6. 4.
^b 1. c. 3. 4.

2. The

2. The ratio of AB to any magnitude CQ, greater than CD, is less than the ratio of M to N.

For, since there may be taken two magnitudes from AB, CD, in the ratio of M to N, so that the remainder of AB shall be less than either DQ or DL, (CL being equal to AB) let these be AG, CH; GB will therefore be less than either HQ or HL, and consequently AG greater than CH. Let $AR : CQ :: AG : CH$, then $GR : HQ :: AG : CH$ ^c. Hence GR is greater than HQ ^d, and therefore greater than GB, which is less than HQ. Consequently AR is greater than AB, and the ratio AB : CQ less than $AR : CQ$ ^b, or $AG : CH$ or M : N.

Consequently the ratio of AB to CD is equal to the ratio of M to N.

O T H E R W I S E,

If magnitudes may be taken from the first and second, in the ratio of the third to the fourth, so that either remainder shall be less than any magnitude proposed, the four magnitudes are proportional.

Fig. 98.

If AB be not to CD, as M to N, let $AR : CD :: M : N$. And,

1. Let AR be less than AB. Then, since there may be taken two magnitudes from AB, CD in the ratio

ratio of M to N, so that the remainder of AB shall be less than RB, let these be AE CF. AE, therefore, is greater than AR. But $AE : CF = M : N = AR : CD$. Wherefore CF is greater than CD^a, which is contrary to hypothesis. Consequently no magnitude less than AB is to CD in the ratio of M to N. And, in like manner, no magnitude less than CD is to AB in the ratio of M to N.

^a 4. 4.
6. 4.

2. Let AR be greater than AB, and let $AB : CO = AR : CD$, then CO will be less than CD^b. But, since $AB : CO = AR : CD = M : N$, by inversion $CO : AB :: N : M$, which is contrary to what has been demonstrated. Consequently no magnitude greater than AB is to CD in the ratio of M to N.

^b 6. 4.

COR. If magnitudes may be taken from the first and third, proportional to the second and fourth, so that either remainder shall be less than any magnitude proposed, the four magnitudes are proportional,

ELEMENTS

ELEMENTS

OF

GEOMETRY.

BOOK V.

DEFINITIONS.

1. **T**WO rectilineal figures are said to be similar, when all the angles of the one are equal to the angles of the other, each to each, in order; and the sides about the equal angles are proportional in the same order.

2. Two magnitudes, and other two, are said to be reciprocally proportional, when one of the former is to one of the latter, as the remaining one of the latter to the remaining one of the former.

3. A straight line is said to be divided in extreme and mean proportion, when the whole is to the greater segment as the greater segment to the less.

4. A

4. A straight line is said to be harmonically divided when it is divided into three segments, so that the whole is to one extreme as the other extreme to the middle segment.

5. Two straight lines are said to be similarly divided when any two corresponding segments of them have the same ratio as the lines themselves.

6. The sum of all the sides of any rectilineal figure is called its perimeter.

A X I O M.

The circumference of a circle is greater than the perimeter of its inscribed polygon, but less than the perimeter of its circumscribing polygon.

P R O P O S I T I O N I.

Triangles between the same parallels are to one another as their bases.

Let ABC, DEF be two triangles between the same parallels AD, BF, the triangle $ABC : DEF :: BC : EF$. Fig. 99.

For, let EG be any part whatever of the base EF less than BC, and GH, HK, KF, the remaining parts
of

of the same. Also, let the lines BL, LM, MN, NO, OP, be taken in the base BC, each equal to EG, and as many as the line BC contains. From A and D let straight lines be drawn to the several points of division in the bases BC, EF. Then, because the bases BL, LM, &c. EG, GH, &c. are all equal, the triangles ABL, ALM, &c. DEG, DGH, &c. are also equal^a. Hence DEG is the same part of the triangle DEF that EG is of the base EF, and the triangle ABC contains DEG just as often as BC contains EF.
^a 2. 2. Therefore $ABC : DEF :: BC : EF$ ^b.
^b def. 4. 4.

COR. 1. Triangles having the same or equal altitudes, are to one another as their bases ; and conversely.

COR. 2. Parallelograms between the same parallels, or having equal altitudes, are to one another as their bases ; and conversely.

COR. 3. Rectangles, and consequently triangles, which have equal bases, are to one another as their altitudes ; and conversely.

P R O P O S I T I O N II.

If a straight line be parallel to one of the sides of a triangle, it cuts the other sides proportionally ; and conversely.

Let

Let the straight line DE be parallel to BC, one of the sides of the triangle ABC, and cut the other sides AB, AC, or them produced in D, E, $AD : DB :: AE : EC$.

For BE, DC being drawn, the triangles DEB, EDC, upon the same base, and between the same parallels, are equal^a; therefore the triangle ADE has the same ratio to each of them^b, $ADE : DEB :: ADE : EDC$. But the former ratio is the same with that of the bases AD, DB^c, because the triangles have the same altitude, and the latter is the same with that of the bases AE, EC^c, for the same reason. Therefore $AD : DB :: AE : EC$ ^d.

Conversely, let $AD : DB :: AE : EC$, then DE is parallel to BC.

For, since $AD : DB :: ADE : DEB$ ^a, and $AE : EC :: ADE : EDC$ ^c, $ADE : DEB :: ADE : EDC$ ^d; therefore the triangles DEB, EDC are equal^e, and consequently between the same parallels^f, that is, DE is parallel to BC.

COR. 1. Straight lines, which meet three parallel straight lines, are cut proportionally.

COR. 2. If two straight lines be cut by any number of parallel lines, the segments intercepted by two of the parallels, have the same ratio as the segments intercepted by any other two.

COR. 3. If any number of straight lines be parallel to the base of a triangle, the segments intercepted by

Q

any

any two of the parallels will have the same ratio as the two sides of the triangle; and these sides will be similarly divided.

COR. 4. Hence the method of dividing a given straight line similarly to a given divided line—of dividing a given straight line in the ratio of two given lines—of dividing a given straight line into any proposed number of equal parts—of finding a fourth proportional to three given straight lines—and of finding a third proportional to two given straight lines.

PROPOSITION III.

If a straight line bisect the vertical angle of a triangle, or the angle adjacent to it, and meet the base, the segments of the base will be directly proportional to the other two sides of the triangle; and conversely.

Fig. 102. Let the vertical angle ABC, or the angle *a*BC adjacent to it, be bisected by the line DB meeting the base AC in D; $AD : DC :: AB : BC$.

For, through C let CE be drawn parallel to BD, to meet AB in E. Because BD is parallel to a side of the triangle ABC, it cuts the other sides proportionally. Thus, $AD : DC :: AB : BE$ ^a. But BE is equal to BC, because the angle BEC = ABD^b = DBC = BCE^b. Therefore $AD : DC :: AB : BC$.

Con-

Conversely, If $AD : DC :: AB : BC$, the angle ABC , or its adjacent angle, will be bisected by the line BD .

For the same construction remaining, $AD : DC :: AB : BE^a$; therefore $AB : BC :: AB : BE^c$. Whence $BE = BC^d$, and the angle $BEC = BCE^e$. Consequently the angle $ABD = BEC^b = BCE = DBC$.

^a 2. 5.
^c ax. 6. 4.
^d 1. c. 4. 4.
^e cor. 4. 1.
^b 16. 1.

COR. If the two vertical angles of any triangle be bisected by straight lines meeting the base, the greatest segment of the base is harmonically divided by the intermediate lines.

PROPOSITION IV.

If two triangles have two angles of the one equal to two angles of the other, each to each, they are similar, and have the homologous sides opposite to equal angles.

Let ABC , DEF be two triangles, having the angle $A = D$, $B = E$, and consequently $C = F$. The sides about any two equal angles shall be proportional, so that the homologous terms shall subtend equal angles, $BA : AC :: ED : DF$.

Fig. 103.

For, if the sides AB , DE be equal, the triangles ABC , DEF are equal in every respect^a, and therefore similar^b. If not, let $AG = DE$, and GH parallel to BC . The triangles AGH , DEF , having the angle

^a 5. 1.
^b ax. 5. 4.
def. 1. 5.

^c 16. 1. angle $A = D$, the angle $G (= B^c) = E$, and the side
^a 5. 1. $AG = DE$ are equal in every respect^a. Thus, AH
 $= DF$. But, since GH is parallel to BC , $BA : AC$
^d 3. c. 2. 5. $:: AG : AH^d$, therefore $BA : AC :: ED : DF$. In
 like manner, $AC : CB :: DF : FE$, and $CB : BA ::$
 $FE : ED$.

COR. 1. A straight line, parallel to one of the sides
 of a triangle, cuts off another triangle similar to it.

COR. 2. The segments of two parallel straight lines,
 intercepted by the two sides of a triangle, have to
 each other the same ratio, as the segments of one of
 these sides intercepted by the parallels from the ver-
 tex.

COR. 3. The segments of two parallel straight lines,
 intercepted by the two sides of a triangle, are cut
 proportionally by any straight line drawn to meet
 them from the vertex.

PROPOSITION V.

*If two triangles have one angle of the one equal to one
 angle of the other, and the sides about the equal angles
 proportional, the triangles shall be similar.*

Fig. 103. Let ABC, DEF be two triangles, having the angle
 $A = D$, and $BA : AC :: ED : DF$, ABC shall be si-
 milar to DEF .

For,

For, if the side $BA = ED$, then $AC = DF^a$, and the triangles are equal in every respect^b. If not, let $AG = DE$, and GH parallel to BC . Then $AG : AH$ ($:: BA : AC$) $ED : DF$, but $AG = ED$, therefore $AH = DF^a$. Hence the triangles AGH , DEF are equal in every respect^b, and consequently the triangles ABC , DEF similar^c. ^a 6. 4. ^b 4. 2. ^c 1. c. 4. 5.

COR. Two isosceles triangles, which have equal vertical angles, are similar.

PROPOSITION VI.

If two triangles have the three sides of the one directly proportional to the three sides of the other, the triangles shall be similar.

Let the two triangles ABC , DEF have the sides BA , AC , CB , of the one, directly proportional to the sides ED , DF , FE , of the other, $BA : AC :: ED : DF$, and $AC : CB :: DF : FE$; ABC , DEF shall be similar. Fig. 103.

For, if the side $BA = ED$, then $AC = DF^a$, $CB = FE^a$, and the triangles equal in every respect^b. If not, let $AG = DE$, and GH parallel to BC . Then $AG : AH$ ($:: BA : AC$) $:: ED : DF$, but $AG = ED$, therefore $AH = DF^a$. Also $AH : HG$ ($:: AC : CB$) $:: DF : FE$, but $AH = DF$, therefore $HG = FE^a$.

Hence

Hence the triangles AGH, DEF are equal in every
 * 6. 1. respect^b, and consequently the triangles ABC, DEF
 * 1. c. 4. 5. similar^c.

PROPOSITION VII.

If two triangles have two sides of the one proportional to two sides of the other, the angles subtended by two homologous sides equal, and the angles opposite to the other two homologous sides, both obtuse, both acute, or one of them right, these triangles shall be similar.

Fig. 103.

Let the two triangles ABC, DEF have the sides BA, AC proportional to ED, DF, viz. $BA : AC :: ED : DF$, the angles B, E opposite to the homologous sides AC, DF equal, and the angles C, F opposite to the other homologous sides AB, DE both obtuse, both acute, or one of them right; the triangles ABC, DEF shall be similar.

* 6. 4.

* 23. 1.

For, if $BA = ED$, then $AC = DF$ ^a, and the triangles equal in every respect^b. If not, let $AG = DE$, and GH parallel to BC. Then $AG : AH (:: BA : AC) :: ED : DF$, but $AG = ED$, therefore $AH = DF$ ^a. Thus the triangles AGH, DEF have two sides of the one GA, AH, equal to two sides of the other, ED, DF, each to each, the angle G (= B = E, and the angle H (= C) and F both obtuse, both
 acute,

acute, or one of them right; therefore these triangles are equal in every respect^b, and consequently the triangles ABC, DEF similar^c. ^{b 23. 1.} ^{c 1. c. 4. 5.}

PROPOSITION VIII.

If two parallelograms, which have one angle of the one equal to one angle of the other, be equal to one another, the sides about the equal angles are reciprocally proportional; and conversely.

Let the two parallelograms AC, GE, having the angle ABC = EBG, be equal to one another, the sides AB, BC, and the sides EB, BG, are reciprocally proportional, $AB : BG :: EB : BC$. Fig. 104.

For, let AB, BG, be in the same straight line, and EB, BC will also be in one straight line^a. Then the parallelogram CG being completed, because the parallelograms AC, GE are equal, $AC : CG :: GE : CG$ ^b. But $AC : CG :: AB : BG$ ^c, and $GE : CG :: EB : BC$ ^c. Therefore $AB : BG :: EB : BC$ ^d. ^a cor. 12. 1.
^b ax. 7. 4.
^c 2. c. 1. 5.
^d ax. 6. 4.

Conversely, If the parallelograms AC, GE, having the angle ABC = EBG, have the sides about these angles reciprocally proportional, they are equal to one another.

For, by hypothesis, $AB : BG :: EB : BC$. But $AB : BG :: AC : CG$ ^e, and $EB : BC :: GE : CG$ ^e.

There-

^d ax. 6. 4. Therefore $AC : CG :: GE : CG^d$, and consequently
^e 1. c. 4. 4. $AC = GE^e$.

COR. 1. If two triangles, which have one angle of the one equal to one angle of the other, be equal to one another, the sides about the equal angles are reciprocally proportional; and conversely.

COR. 2. Equal rectangles have their sides reciprocally proportional; and conversely.

COR. 3. If four straight lines be proportional, the rectangle under the extremes is equal to the rectangle under the means; and conversely.

COR. 4. If a straight line be divided into three segments, so that the rectangle under the whole line, and middle segment, be equal to the rectangle under the extreme segments, it is divided harmonically; and conversely.

COR. 5. If three straight lines be proportional, the rectangle under the extremes is equal to the square of the mean; and conversely.

COR. 6. If a straight line be so divided, that the rectangle under the whole, and one of the segments be equal to the square of the other, it is divided in extreme and mean proportion; and conversely.

COR. 7. Equal triangles, or equal parallelograms, have their bases and altitudes reciprocally proportional; and conversely.

PROPO.

PROPOSITION IX.

Two parallelograms, which have one angle of the one equal to one angle of the other, are to each other in a ratio compounded of the ratios of the sides of the one to the sides of the other, each to each, about the equal angles.

Let AC, GE be two parallelograms, having the angle $ABC = EBG$, AC is to GE, in a ratio compounded of the ratios of AB to BG, and CB to BE. Fig. 104.

For the same construction remaining, as in the last proposition, $AC : GE$ is the ratio that is said to be compounded of the ratios $AC : CG$, and $CG : GE$ ^a.
 But $AC : CG = AB : BG$ ^b, and $CG : GE = CB : BE$ ^b.
 Therefore the ratio $AC : GE$ is the same with that which is compounded of the ratios $AB : BG$, and $CB : BE$ ^c.

COR. 1. Two triangles, which have one angle equal to one angle, are to each other in a ratio compounded of the ratios of the sides of the one to the sides of the other, each to each, about the equal angles.

COR. 2. Any two rectangles are to each other in the compound ratio of their sides—and any ratio compounded of two ratios, (whose terms are straight lines,)

R

lines,)

lines,) is the same with the ratio of the rectangles under their homologous terms.

COR. 3. Two triangles, which have one angle equal to one angle, are to each other as the rectangles under the sides about the equal angles.

COR. 4. The rectangles under the corresponding terms of two analogies are proportional.

COR. 5. If four straight lines be proportional, their squares are also proportional; and conversely.

P R O P O S I T I O N X.

Two similar triangles are to each other in the duplicate ratio of their homologous sides.

Fig. 105.

Let BAC, EDF be two similar triangles, having the angle $\angle ABC = \angle DEF$, and $AB : BC :: DE : EF$; the triangle BAC is to the triangle EDF, in the duplicate ratio of BC to EF.

For, let GH be a third proportional to BC, EF, so
^a 4. c. 2. 5. that $BC : EF = EF : GH^a$. Then, because the angles at B, E are equal, the ratio BAC : EDF is the same with that which is compounded of $BC : EF$, and
^b 1. c. 9. 5. $AB : DE^b$. But, since $AB : BC :: DE : EF$, the
^c 7. 4. ratio $AB : DE = BC : EF^c = EF : GH$. Therefore the ratio BAC : EDF is the same with that which is
^d cor. 17. compounded of $BC : EF$, and $EF : GH^d$, that is, the
 4. ratio

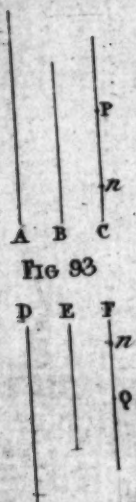


Fig 93

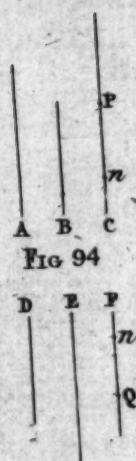


Fig 94

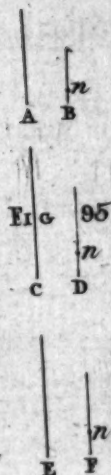


Fig 95

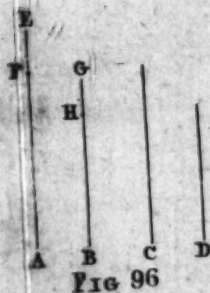


Fig 96

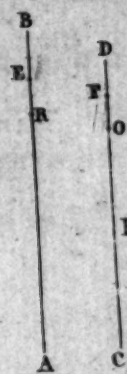


Fig. 97.

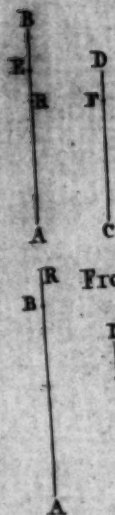
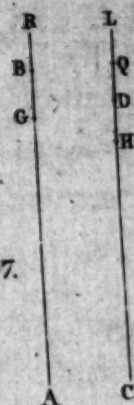


Fig. 98.

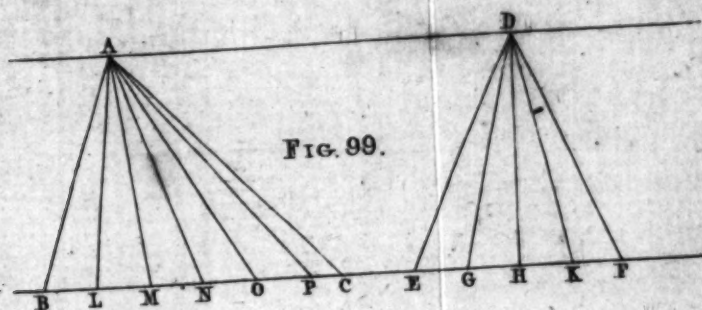


Fig. 99.

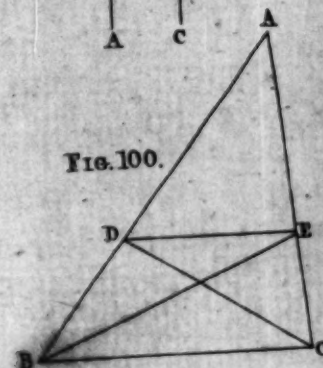


Fig. 100.

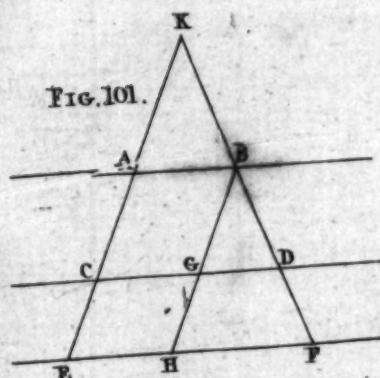


Fig. 101.

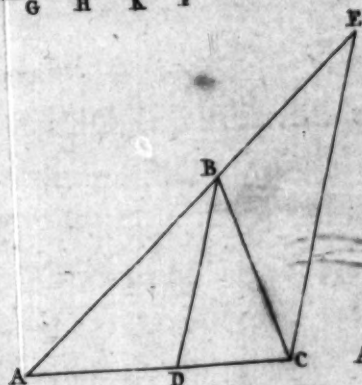


Fig. 102.

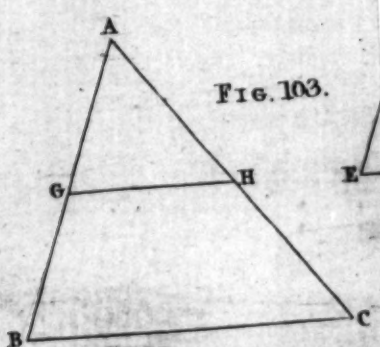
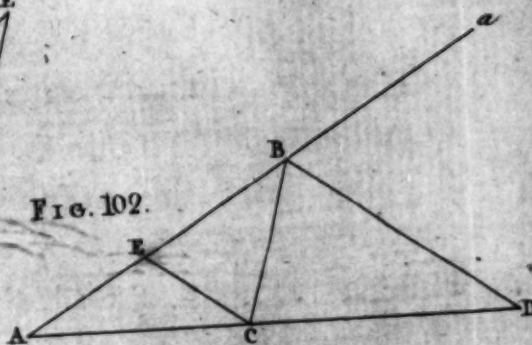


Fig. 103.

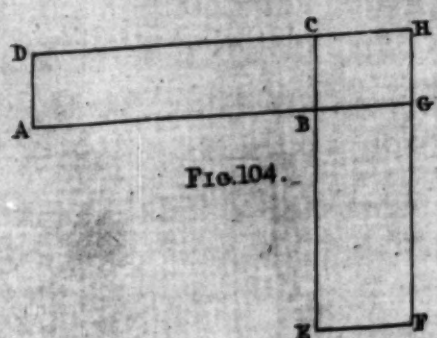
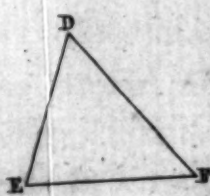


Fig. 104.

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ratio $BAC : EDF$ is the same with the ratio $BC : GH$, which is the duplicate ratio of BC to EF ^c. def. 10.
4

COR. 1. Two squares, or, in general, two similar parallelograms, are to each other in the duplicate ratio of their homologous sides.

COR. 2. Two similar triangles, or two similar parallelograms, are to each other as the squares of their homologous sides.

COR. 3. If three straight lines be proportional, the first is to the third as the square of the first to the square of the second, (or as the square of the second to the square of the third); and conversely.

PROPOSITION XI.

Upon a given straight line to describe a rectilineal figure similar to a given one, and similarly situated.

Let $ABCDE$ be the given rectilineal figure described upon AB , and FG the given straight line upon which it is required to describe a figure similar to the given one, and similarly situated. Let the angle FGH be made equal to ABC ^a, and let GH be a fourth proportional to AB , BC , FG ^b. In like manner, let the angle $GHK = BCD$, and HK a fourth proportional to BC , CD , GH ; the angle $HKL = CDE$, and KL a fourth proportional to CD , DE , HK ; and let

Fig. 106.
13. 1.
4. c. 2. 5.

let FL be joined ; the figure FGHL will be similar to ABCDE.

For, since the angles B, G are equal, and $AB : BC :: FG : GH$, the triangles ABC, FGH are similar ^c. Hence the angle ACB = FHG, but BCD = GHK, therefore ACD = FHK. Hence also $AC : BC :: FH : GH$, but $BC : CD :: GH : HK$; therefore, by equality, $AC : CD :: FH : HK$ ^d. Thus, the triangles ACD, FHK, having the angles at C and H equal, and $AC : CD :: FH : HK$, are similar ^c. In the same manner, ADE, FKL are proved similar. Hence the angles at E and L are equal, and $DE : EA :: KL : LF$. But the whole angles at A and F are equal, because they are composed of angles that are equal, each to each, and, since the lines AE, AD, AC, AB, from the similarity of the triangles, are directly proportional to FL, FK, FH, FG, by equality $AE : AB :: FL : FG$ ^d. Therefore the rectilineal figures ABCDE and FGHL are similar ^c.

^c def. 1. 5.

COR. 1. Similar polygons may be divided into the same number of triangles similar each to each, and having the same ratio each to each that the polygons have.

COR. 2. Those polygons, which consist of the same number of triangles, similar each to each, and similarly situated, are similar.

COR. 3. Similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

COR.

COR. 4. Similar rectilinear figures are to one another as the squares of their homologous sides.

COR. 5. If three straight lines be proportional, the first is to the third as a rectilinear figure described upon the first to a similar rectilinear figure described upon the second.

COR. 6. If two rectilinear figures of the same number of sides have as many angles wanting two, of the one, equal to the corresponding angles of the other, each to each, and the sides about the equal angles proportional in the same order, the two figures are similar; and, if the sides be equal, each to each, the figures are both equal and similar.

PROPOSITION XII.

The perpendicular from the right angle, upon the hypotenuse of a right angled triangle, divides the triangle into two others, similar to the whole and to one another.

Let ABC be the right angled triangle, and BD the perpendicular upon the hypotenuse, the triangles ABD, DBC are similar to the whole, and to one another. Fig. 107.

For, since the triangles ABD, ABC have the angle at A common to both, and the angles at D, B equal,

as

4 5. as being right angles, they are similar^a. In like manner, the triangles DBC, ABC are similar. Consequently the triangles ABD, DBC are similar to one another^b.

^b ax. 2. 1.

ax. 6. 4.

COR. 1. The vertical angles are alternately equal to the angles at the base.

COR. 2. Each side is a mean proportional between the hypotenuse and the adjacent segment; and the square of the former is equal to the rectangle of the latter.

COR. 3. The perpendicular is a mean proportional between the segments of the hypotenuse.

COR. 4. A perpendicular to the diameter of a circle, from any point in the circumference, is a mean proportional between the segments of the diameter.

COR. 5. Hence the method of finding a mean proportional between two given straight lines is obvious.

P R O P O S I T I O N XIII.

The rectangle under the two sides of a triangle is equal to the rectangle under the perpendicular, and the diameter of the circumscribing circle.

Fig. 108.

Let ABC be a triangle, of which BD is the perpendicular on the base AC, and let BE be a diameter of

of the circumscribing circle, the rectangle under AB, BC is equal to the rectangle under BD, BE.

For AE being joined, the triangles ABE, CBD have the angles at E, C equal ^a, because they stand upon the same arch, and the angles at A, D equal, because they are right angles ^b. Wherefore these triangles are similar ^c, and $AB : BE :: DB : BC$. Consequently the rectangle under AB, BC is equal to the rectangle under BD, BE ^d.

^a 1. c. 7. 3.

^b 9. 3.

^c 4. 5.

^d 3. c. 8. 5.

PROPOSITION XIV.

If a straight line bisect the vertical angle of a triangle, and meet the base, the square of that line, together with the rectangle under the segments of the base, is equal to the rectangle under the two sides of the triangle.

Let the vertical angle ABC, of the triangle BAC, Fig. 109. be bisected by the line BD meeting the base in D, the square of BD, together with the rectangle ADC, shall be equal to the rectangle under AB, BC.

For, let a circle be described about the triangle, and BD being produced, to meet the circumference in E, let A, E be joined. The triangles ABE, CBD, having the angles at B equal by hypothesis, and the angles at E, C equal, as standing upon the same arch, are similar ^a. Therefore $EB : BA :: CB : BD$, and ^a 4. 5. the

the rectangle EBD equal to the rectangle under AB, ^b 3. c. 8. 5. BC ^b. But the former rectangle is also equal to the ^c 2. c. 6. 2. square of BD, added to the rectangle EDB ^c, or ^d 17. 3. ADC ^d. Consequently the square of BD, added to the rectangle ADC, is equal to the rectangle under AB, BC.

COR. If a straight line bisect the angle adjacent to the vertical angle of a triangle, and meet the base produced, the difference between the square of that line, and the rectangle under the segments of the base, is equal to the rectangle under the two sides of the triangle.

PROPOSITION XV.

The rectangle under the two diagonals of any quadrilateral figure, inscribed in a circle, is equal to the sum of the rectangles under the opposite sides.

Fig. 110. Let ABCD be a quadrilateral figure, inscribed in a circle, and AC, BD its diagonals; the rectangle under AC, BD is equal to the sum of the rectangles under AB, CD, and AD, BC.

For, from the point B, let BE be drawn to make with AB the angle ABE equal to CBD ^a, and to meet AC in E. The triangles BCD, BEA, having the angles at B equal, and likewise the angles at D and

A equal, are similar ^b. Therefore $BD : DC :: BA$ ^{b 4. 5.}
 $: AE$, and the rectangle under BD , AE equal to the
 rectangle under AB , CD ^c. But the triangles DBA , ^{c 3. c. 8. 5.}
 CBE have also the angles at B equal, and the angles
 at C and D equal. Therefore $BD : DA :: BC : CE$,
 and the rectangle under BD , EC equal to the rectangle
 under AD , BC . Consequently the sum of the rec-
 tangles BD , AE and BD , EC , that is, the rectangle
 AC , BD ^d, is equal to the sum of the rectangles AB , ^{d 6. 2.}
 CD , and AD , BC .

PROPOSITION XVI.

*If three straight lines be proportional, the first is to the
 third, as the square of the difference of the first and
 second to the square of the difference of the second and
 third; and conversely.*

Let $AB : BC :: BC : BD$, AC , the difference of the Fig. III.
 first and second, and CD the difference of the second
 and third; then $AB : BD :: AC^2 : CD^2$.

For, let the semicircle BEA be described upon AB ,
 let DE , perpendicular to AB , meet the circumference
 in E , and let EB , EC , and EA be drawn. Then
 the squares of BE , BC , being each equal to the rec-
 tangle ABD ^a, are equal to one another, and the side ^{a 2. c. 12. 5.}
 BE equal to the side BC ^b. Hence the angle BEC is ^{& 5. c. 8. 5.}
 equal ^{b 1. c. 4. 2.}

equal to BCE, or to the sum of CAE, CEA, and taking away the equal angles BED, CAE, there remains the angle DEC equal to CEA. Thus, the angle AED is bisected by EC; therefore $AC : CD :: AE : ED^c$. But $AE : ED :: AB : BE^d$, or BC. Therefore also, $AB : BC :: AC : CD$, and $AB^2 : BC^2 :: AC^2 : CD^2^e$. Consequently $AB : BD :: AC^2 : CD^2^f$.

^c 3. 5.
^d 12. 5.

^e 5. c. 9. 5.

^f 3. c. 10. 5.

Conversely, If $AB : BD :: AC^2 : CD^2$, then $AB : BC :: BC : BD$. For, the same construction remaining, $AB : BD :: AB^2 : BE^2^f$; therefore $AC^2 : CD^2 :: AB^2 : BE^2$, and $AC : CD :: AB : BE^c :: AE : ED^d$. Hence the angle AED is bisected by EC. But the angle BED is equal to CAE. Therefore the angle BEC is equal to the sum of the angles CEA, CAE, that is, to the angle BCE. Consequently BC is equal to BE, and $AB : BC :: BC : BD^a$.

^a 2. c. 12. 5.
& 5. c. 8. 5.

COR. If three straight lines be proportional, the first is to the third as the square of the sum of the first and second to the square of the sum of the second and third.

PROPOSITION XVII.

To find a point in a given straight line, or in that line produced, so that the rectangle under the segments may be equal to the rectangle under two given lines.

Let

Let AB be the given straight line, in which it is required to find a point, such, that the rectangle under the segments shall be equal to a given rectangle. Let AC, BD, be perpendicular to AB, and equal to the sides of the given rectangle, each to each, Let a semicircle be described upon AB, and let the straight line that joins the points C, D, meet its circumference in E. Then the perpendicular to CD from E will meet AB in the point F required. Fig. 112.

For, let AE, EB be joined. Then, because DEF, DBF are right angles, the angle EFA is equal to the angle EDB^a, and because AEB^b, FED are right angles, the angle AEF is equal to the angle DEB. Therefore the triangles EAF, EBD are similar^c. In like manner, the triangles EBF, EAC are similar.

From the former $AF : AE :: BD : BE$.

From the latter $AE : AC :: BE : FB$.

Therefore, by equality^d $AF : AC :: BD : FB$.

Consequently the rectangle AFB is equal to the rectangle under AC, BD^e.

PROPOSITION XVIII.

To find a point in a given straight line, or in that line produced, so that the square of one of the segments may be equal to the rectangle under the other segment and a given line.

Let

^a 10. 3.
^b 2. c. ib.
^c 1. c. ib.
^d 1. c. 7. 3.
^e 9. 3.
^f 4. 5.

^d 9. 4.

^e 3. c. 8. 5.

Fig. 113.

Let AB be the given straight line, in which it is required to find a point, such, that the square of one of the segments shall be equal to the rectangle under the other segment, and the given line M . Let AB be produced to C , so that $BC = M$, and upon AC let the semicircle ADC be described. Let BD be perpendicular to AC , BC bisected in E , and E, D joined. Then a circle described about the center E , with the radius ED , will intersect AB in the point F required.

cor. 8. 2

9. 2.

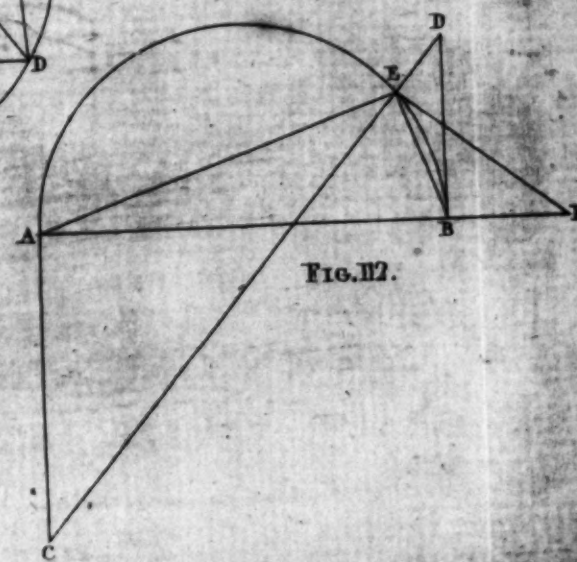
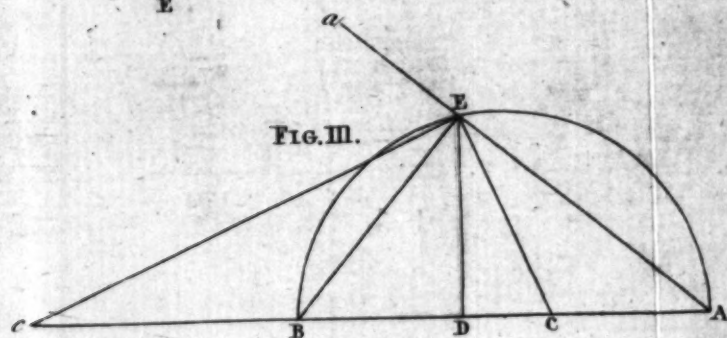
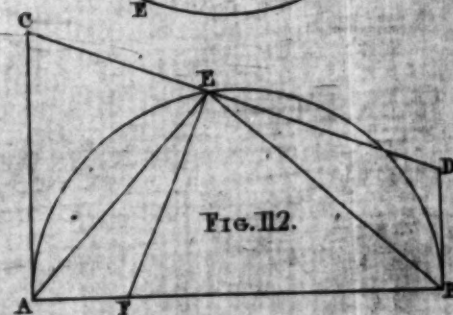
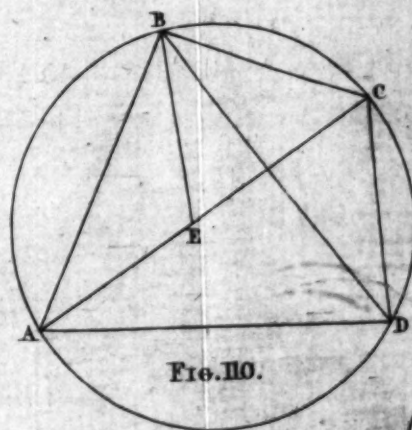
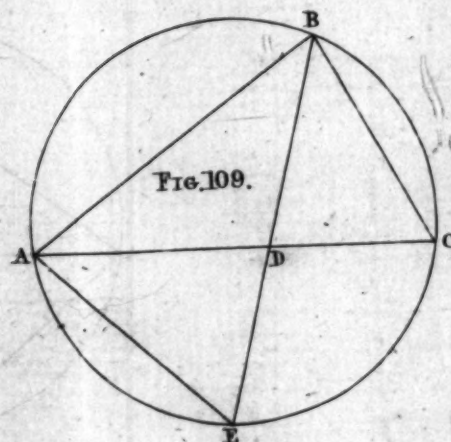
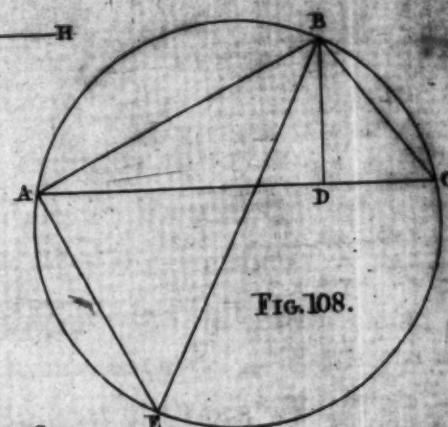
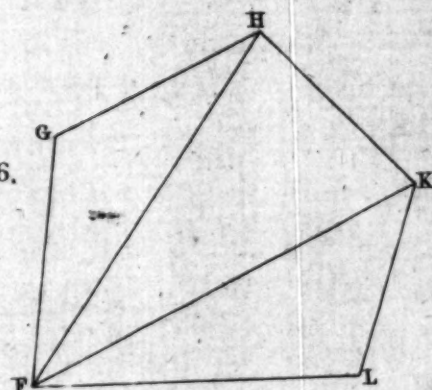
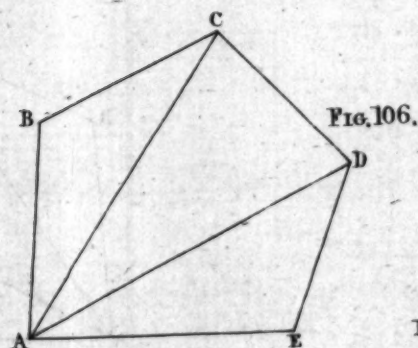
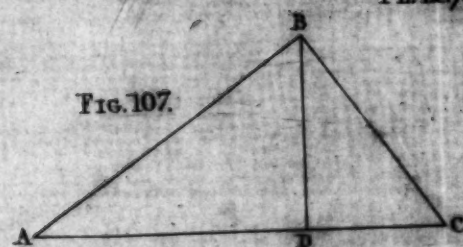
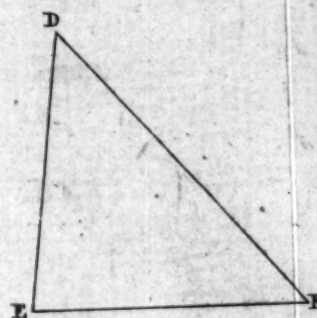
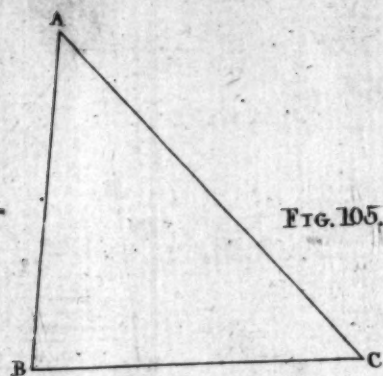
c. 3. 17. 3.

For, since BC is bisected in E , the rectangle CFB , together with the square of BE , is equal to the square of EF ^a, or to the sum of the squares of BD, BE ^b; and taking away the common square of BE , there remains the rectangle CFB equal to the square of BD . But the square of BD is equal to the rectangle ABC ^c. Therefore the rectangle CFB is equal to the rectangle ABC . Hence, by adding or taking away the common rectangle CBF , we shall have the square of FB equal to the rectangle under AF and BC .

PROPOSITION XIX.

In the same circle, or in equal circles, angles at the center are to one another as the arches upon which they stand.

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Let ABC , DEF , be any two angles, at the centers of the equal circles AGC , DHF ; the angle $ABC : DEF :: AC : DF$.

For, let Dn be any part whatever of the arch DF , less than AC , and let Am , mo , oq , qr , be the arches which AC contains, each equal to Dn . The angles ABm , mBo , oBq , qBr , which stand upon these arches, are each equal to DEn . Wherefore the angle ABC contains DEn just as often as the arch AC contains Dn . But DEn , it is evident, is the same part of the angle DEF , that Dn is of the arch DF . Consequently the angle $ABC : DEF :: AC : DF$.

COR. 1. In the same, or equal circles, angles at the circumference, are to one another as the arches upon which they stand.

COR. 2. An angle at the center of a circle has the same ratio to four right angles, that the arch upon which it stands has to the whole circumference.

COR. 3. Sectors of the same, or equal circles, are to one another as their arches. For this may be proved in the very same manner.

PROPOSITION XX.

Circles are to one another in the duplicate ratio of their diameters.

Let

Let ABC , EFG be two circles, of which AC , EG are diameters, and let $AC : EG :: EG : RQ$, the circle $ABC : EFG :: AC : RQ$.

For, let the squares $ABCD$, $EFGH$ be inscribed in the circles. Let the arches which the sides of these squares subtend be bisected, and the regular polygons KL , MN be formed by joining each point of section to the adjacent angular points of the square. Let the arches which the sides of these polygons subtend be bisected, and other polygons formed, by joining each point of section to the adjacent angular points of the former; and so on.

Any one of these polygons, in the circle ABC , as KL , is to the polygon MN , of the same number of sides, in the circle EFG , as AC to RQ . For, since each of the polygons KL , MN consists of the same number of isosceles triangles, and magnitudes have the same ratio as their like parts, $KL : MN :: AOL : EPN$. But the isosceles triangles AOL , EPN , having equal vertical angles, (each of them being the same part of four right angles^a, are similar^b. Therefore $AOL : EPN (:: AO : \frac{1}{2}RQ^c) :: AC : RQ$.

Consequently $KL : MN :: AC : RQ^d$.

Again, because the difference between one of the circles and its inscribed square $ABCD$ is the sum of the four segments AB , BC , CD , DA , and each of the triangles inscribed in these segments is greater than half its segment, (being equal to half the circumscri-

^a 5. c. 2. 3.

^b cor. 5. 5.

^c 10. 5.

^d 1. 4.

bing parallelogram), by taking the polygon KL from the circle ABC, there will be taken away more than half that difference. In like manner, by taking the next polygon from the same circle, there will be taken away more than the half of what last remained; and so on. Therefore, from the two circles ABC, EFG, there may be taken away two polygons in the ratio of AC to RQ, so that either of them shall leave a remainder less than any magnitude proposed. • lem. 3.

Consequently the circles themselves are in the same ratio. 4.
f 18. 4.

COR. 1. Similar sectors (that is, sectors of equal angles,) are to one another in the duplicate ratio of their radii.

COR. 2. Circles, and similar sectors of these circles, are to one another as the squares of their diameters or radii.

PROPOSITION XXI.

Every circle is equal to a rectangle under the radius, and a straight line, equal to half the circumference.

Let ABC be a circle, of which O is the centre, and let the straight line BR be equal to half its circumference; the circle ABC shall be equal to the rectangle OBR. Fig. 116.

For,

For, let the square ABCD, and the polygon KL be inscribed in the circle, as in the preceding proposition. Also, let the square EFGH, and polygon MQ, be described about the circle, so that their sides touch the circumference in the angular points of the former.

^a 2. c. 13. Then, because OH bisects the angle DOC^a, it passes
^{3. 6. 1.} through the point K^b, and is perpendicular to MN^c,
^b 1. c. 2. 3. a side of the circumscribing polygon. Hence MH is
^c 11. 3. greater than MK^d or MD^e; therefore the triangle
^d 2. c. 19. MKH is greater than MKD^e, and consequently the
^{1.} triangle MNH greater than half the triangular space
^e 1. c. 2. 2. DHC, without the circle. Thus, the sides of the polygon MQ cut off from the square EG more than half its excess above the circle. In like manner, the next circumscribing polygon cuts off more than half the remainder; and so on. Therefore a polygon may be described about the circle, so that their difference
^f lem. 3. 4. shall be less than any magnitude proposed^f.

Now, if the rectangle OR be greater than the circle ABC, a polygon may be described about the circle that shall differ less from it than OR, that is, that shall be less than OR. Let MQ, then, be a circumscribing polygon less than OR. Because MQ may be divided into isosceles triangles, each of which is equal to the rectangle under the radius of the inscribed
^{2 c. 2. 2]} circle, and half its base^g, the whole polygon is equal
^{6 2.} to the rectangle under the radius of the inscribed circle and half its perimeter^h. But half its perimeter

is greater than BR , half the circumference¹. There- i ax. 5.
fore also that rectangle, or the polygon MQ , is
greater than the rectangle OR , which is contradic-
tory.

Again, if the rectangle OR be less than the circle
 ABC , a polygon may be inscribed in the circle that
shall differ less from it than OR , that is, that shall
be greater than OR . Let KL then be an inscribed
polygon, greater than OR . Then, as before, KL is
equal to the rectangle under the radius of its inscribed
circle, and half its perimeter. But the radius of its
inscribed circle is less than OB , and half its perimeter
is less than BR ¹. Therefore also that rectangle, or
the polygon KL , is less than the rectangle OR , which
is contradictory.

Consequently the rectangle OR is equal to the
circle ABC .

COR. Hence any sector is equal to a rectangle un-
der the radius, and a straight line equal to half its
circumference.

PROPOSITION XXII.

*The circumferences of circles are to one another as their
radii.*

T

Let

Fig. 117. Let ABC , DEF be any two circles, of which OB , HE are radii, the circumference $ABC : DEF :: OB : HE$.

For, let the straight lines BR , EK , touching the circles in the points B , E , be each equal to half the circumference of its circle, and let the rectangles OR , HK , and squares OG , HI be completed. Because the circles are equal to the rectangles OR , HK^a , and have the same ratio as the squares OG , HI^b ; $OR : HK :: OG : HI$, and, by alternation, $OR : OG :: HK : HI$. But rectangles of the same altitude are to one another as their bases; therefore $BR : BG :: EK : EI$, and alternately $BR : EK :: BG : EI$. Consequently $2BR : 2EK :: OB : HE$.

COR. 1. The arches of similar sectors are to one another as their radii.

COR. 2. The arches of any two sectors are to one another, in the ratio compounded of the ratios of their radii and angles.

PROPOSITION XXIII.

If similar rectilineal figures be described upon the three sides of a right angled triangle, the figure upon the hypotenuse is equal to the other two taken together.

Fig. 118. Let ABC be a right angled triangle, of which AC is the hypotenuse, and let AE , FB , BG , be similar rectilineal

rectilineal figures described upon the sides; the figure AE upon the hypotenuse is equal to the sum of the figures FB, BG, upon the other two sides.

For, let BD be perpendicular to AC; then, because $AC : AB :: AB : AD^a$, $AC : AD :: AE : FB^b$.

In like manner, $AC : CD :: AE : BG$.

Therefore $AC : AD + CD :: AE : FB + BG^c$.

But $AC = AD + CD$, consequently $AE = FB + BG^d$.

COR. If circles be described upon the three sides of a right angled triangle, as diameters, or with these sides as radii, the circle of the hypotenuse is equal to the sum of the other two,

A P P E N D I X.

THEOREM 1. If, from two angular points of any triangle, straight lines be drawn, to bisect the opposite sides, they will divide each other into segments, having the ratio of two to one.

THEOR. 2. If two triangles have two angles equal, and other two angles together equal to two right angles, the sides about the remaining angles will be proportional.

THEOR. 3. If, from any point in the circumference of a circle, straight lines be drawn through the extremities of a chord, to meet the diameter that is perpendicular

pendicular to it, the radius will be a mean proportional between the segments of that diameter intercepted from the center.

THEOR. 4. If, from the extremities of the base of a triangle, two straight lines be drawn, each parallel to one of the sides, and equal to the other; then straight lines, joining their other extremes with the other extremes of the base, will cut off equal segments from the sides, and each of these will be a mean proportional between the other two segments.

THEOR. 5. If a regular polygon be described about a circle, and another, of the same number of sides, inscribed in it; then a third regular polygon, of double the number of sides, inscribed in the same circle, will be a mean proportional between the two former.

THEOR. 6. If, from the extremities of the base of a triangle, perpendiculars be let fall upon the opposite sides, and likewise straight lines drawn to bisect the same, the intersection of the perpendiculars, the intersection of the bisecting lines, and the center of the circumscribing circle, will be in the same straight line.

THEOR. 7. If, from a point without a circle, two tangents be drawn, and from the same point, and one of the points of contact, two straight lines be drawn, to meet in the circumference, then shall the chord, which is drawn from the other point of contact, parallel

FIG. 113.

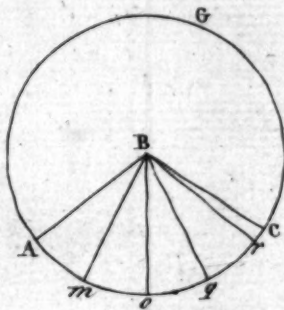
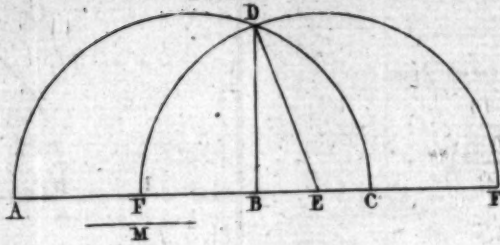


FIG. 114.

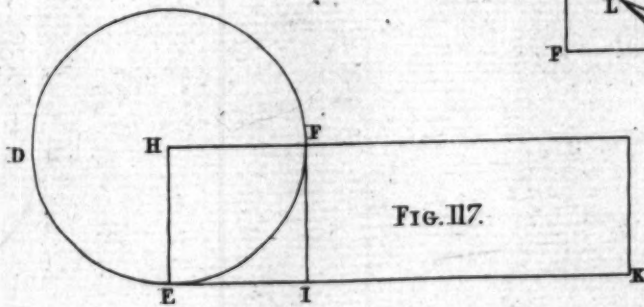
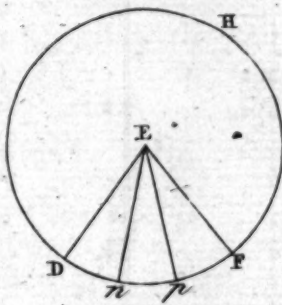


FIG. 117.

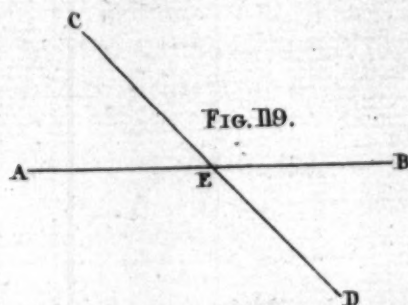


FIG. 119.

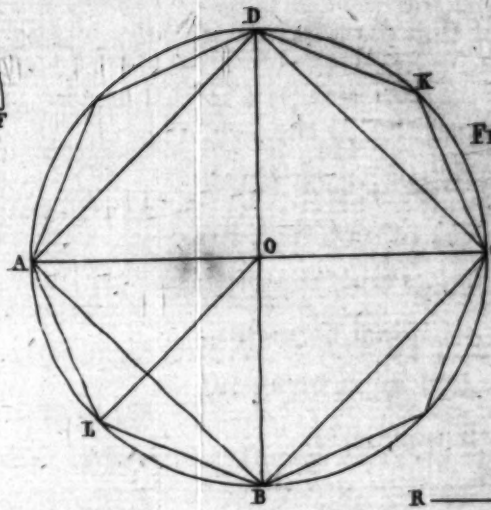


FIG. 121.

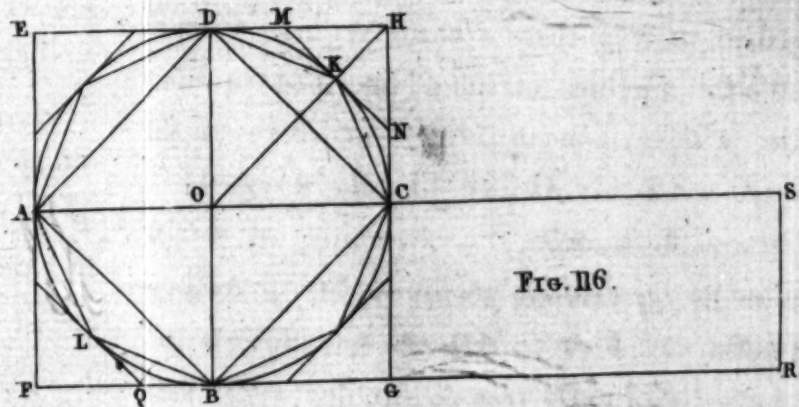
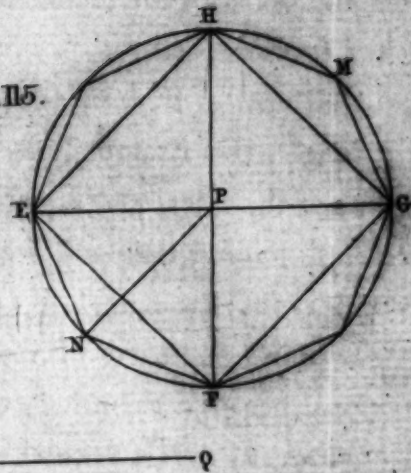


FIG. 124.

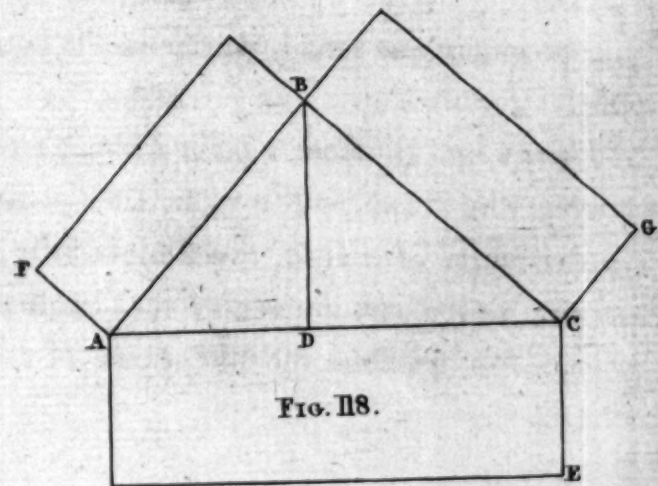


FIG. 126.

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rallel to one of the tangents, be divided by these straight lines, so that the segment intercepted by the latter from the point of contact, will be a mean proportional between the whole chord and the segment intercepted by the former.

THEOR. 8. Let there be two concentric circles, and let a tangent to the less cut the circumference of the greater; from the point of contact let two equal chords be applied in the former, and from the point of section let others parallel to them be applied in the latter; the sum of the chords in the less circle will be equal to the sum or difference of those in the greater, according as these last are on the same, or on different sides of the tangent.

THEOR. 9. Let there be a straight line harmonically divided, and from any point without it let straight lines be drawn through its extremities, and the points of division; then any straight line drawn from the one extremity, to terminate in the line that passes through the other, will also be harmonically divided by the intermediate lines.

THEOR. 10. If, from the extremities of the base of a triangle, straight lines be drawn through the same point in the perpendicular, to intersect the opposite sides, straight lines, joining the points of section with the foot of the perpendicular, will make equal angles with the base.

THEOR.

THEOR. 11. If, from the extremities of the diameter of a circle, two perpendiculars be raised, each equal to the side of the inscribed square, and from their extremities straight lines be drawn through any point in the circumference, to meet that diameter, these straight lines will each intercept a segment from the farthest extremity of the diameter, such that the sum of their squares shall be equal to the square of the diameter.

THEOR. 12. If, upon the base of a triangle, a right angled triangle be constituted, having its hypotenuse equal to the sum of the two sides of the triangle, and through the vertex a straight line be drawn parallel to the base; the segment of the hypotenuse between the parallels, half the sum of the two sides, and half the difference of the segments of the base, will form a right angled triangle.

PROBLEM 1. The perpendicular, the difference of the segments of the base, and either the sum or difference of the two sides of a triangle, are given to construct it.

PROB. 2. The perpendicular, the difference of the angles at the base, and either the sum or difference of the two sides of a triangle, are given to construct it.

PROB. 3. In a quadrilateral figure, inscribed in a circle, two adjacent sides, one of which is equal to the diagonal drawn from their intersection, and the ratio of

of the segments of the other diagonal are given to construct the figure.

PROB. 4. To describe a quadrilateral figure, having its sides of a given magnitude, and its opposite angles together equal to two right angles.

PROB. 5. To divide a given triangle into three others, proportional to three given lines, by straight lines drawn from a point within the triangle.

PROB. 6. From a given triangle to cut off another, having a given ratio to the whole, by a straight line drawn through a given point.

PROB. 7. A circle, and a point within or without it being given, to find another point, such, that any straight line being drawn through it, to intersect the circumference, the distances of the points of section from the given point shall be to one another directly as the segments of the straight line intercepted by the circumference from the point required.

PROB. 8. A circle, and a point without it, being given, to find another point, such, that any straight line being drawn through it, to intersect the circumference, the distances of the points of section from the given point shall be to one another in the duplicate ratio of their distances from the point required.

ELEMENTS

E L E M E N T S

O F

G E O M E T R Y.

B O O K VI.

D E F I N I T I O N S.

1. **A** Straight line is said to be perpendicular to a plane, when it is perpendicular to all the straight lines that can be drawn in that plane from the point on which it insists.

2. The inclination of a straight line to a plane is the acute angle which it forms with the line joining its intersection with the plane and the extremity of a perpendicular to the plane, drawn from any point in the straight line.

3. Two planes are perpendicular to one another, when any straight line drawn in either of the planes, at right angles to their line of common section, is perpendicular to the other.

4. The

4. The inclination of two planes is the angle formed by straight lines drawn in these planes, at right angles to their line of common section, from the same point in it.

5. Two planes, or a straight line and a plane, are parallel, when being produced indefinitely they do not meet.

6. A solid is that which has length, breadth, and thickness.

7. A solid angle is that which is formed by more than two plane angles at the same point, but not in the same plane.

8. Two solids, bounded by planes, are similar, when their solid angles are equal, and their plane figures similar, each to each.

9. A parallelopipedon is a solid bounded by six planes, of which the opposite ones are parallel. If the adjacent planes be perpendicular to one another, it is a rectangular parallelopipedon.

10. A cube is a rectangular parallelopipedon, of which the six sides are squares.

11. A prism is a solid, of which the sides are parallelograms, and the ends are plane rectilineal figures.

12. A pyramid is a solid, of which the sides are triangles, having a common vertex, and the base any plane rectilineal figure. If the base be a triangle, it

is a triangular pyramid. If a square, it is a square pyramid.

13. A cylinder is a solid described by the revolution of a rectangle about one of its sides remaining fixed; which side is named the axis; and either of the circles described by its adjacent sides the base of the cylinder.

14. A cone is a solid described by the revolution of a right angled triangle about one of its sides remaining fixed; which side is named the axis, and the circle described by the other side, the base of the cone.

15. Cones or cylinders are similar, when they are described by similar figures.

PROPOSITION I.

If two straight lines intersect each other, they are both in one plane.

Fig. 119. The two straight lines AB, CD, which intersect each other in the point E, are both in one plane.

For, let any plane meet the straight line CD, in the points E, D; and because CD passes through these
 “def. 5. 1. points, it is wholly in that plane”. Let the same plane always touching the line CD revolve about it, until it meet the point A; then the straight line AB, passing through

through the points A, E, is likewise wholly in that plane^a. Therefore the straight lines AB, CD are in one and the same plane. ^a def. 5. 1.

COR. 1. If two planes intersect each other, their common section is a straight line. For, if two planes pass through the points E, D, the straight line CD is wholly in each of them, and is therefore their line of common section.

COR. 2. Three straight lines, which intersect one another in three points, are all in one plane. Or, a plane may be extended through any three points whatever.

PROPOSITION II.

If a straight line be perpendicular to two straight lines intersecting each other, it is perpendicular to their plane.

Let the straight line EF be perpendicular to each of the straight lines AB, CD, which intersect each other in the point E; EF is perpendicular to their plane. Fig. 120.

For, let any straight line GH be drawn through the point E, in the plane of AB, CD. Let the lines EA, EB, EC, ED, be assumed equal. From any point F in the perpendicular let FA, FB, FC, FD, be drawn; and let AC, BD be joined. The triangles AEC, BED, are equal in every respect^a, for they have two sides of the one equal to two sides of the other, each

to

to each, and the included angles equal; therefore $AC = BD$, and the angle $C = D$. The triangles CGE , EHD , having the angle $C = D$, the angles at E equal, and the side $CE = ED$, will also have the side $CG = DH$, and $GE = EH$ ^b. The right angled triangles FEA , FEB , FEC , FED , have the side FE common to all, and the other sides about the right angles equal, therefore the hypotenuses FA , FB , FC , FD , are equal^a. Thus, the triangles FCA , FBD , have the three sides of the one equal to the three sides of the other, each to each, therefore the angle $FCG = FDH$ ^c. But the triangles FCG , FDH , have also the sides about these angles equal, each to each; therefore the base $FG = FH$ ^a. Hence the triangles FEG , FEH , are mutually equilateral; therefore the angles FEG , FEH are equal^c, and each of them a right angle. Consequently the straight line EF is perpendicular to any straight line drawn in the plane of AB , CD , from the point E , that is, it is perpendicular to the plane of these lines^d.

Fig. 121.

COR. 1. There cannot be more than one straight line perpendicular to a plane at the same point in it. For, if the lines FE , EK be both perpendicular to the plane AB at the point E , let their plane meet AB in the line of common section GEH , and the lines FE , EK are both perpendicular to GH , which is impossible.

COR.

COR. 2. There cannot be more than one straight line perpendicular to a plane from the same point without it. For, if the lines FE, FG be both perpendicular to the plane AB from the point F, let their plane meet AB in the line of common section EG, and the angles FEG, FGE, are both right angles; which is impossible. Fig. 122.

COR. 3. There cannot be more than one plane perpendicular to a straight line at the same point. For, if each of the planes CD, EF be perpendicular to the line AB, at the point B, let a plane touching AB intersect them both in the lines of common section BC, BE, then ABC, ABE are in one plane, and both right angles; which is impossible. Fig. 123.

COR. 4. If a straight line be perpendicular to three or more straight lines at the same point, they are all in one and the same plane.

PROPOSITION III.

Every straight line parallel to a straight line that is perpendicular to a plane, is perpendicular to the same; and conversely.

Let the straight line AB be perpendicular to the plane FG, then any straight line CD, parallel to AB, will also be perpendicular to FG. Fig. 124.

For.

For, let the plane of the parallels meet FG in the line BD; let DE be perpendicular to BD, in the plane FG, and equal to AB, and let AD, AE, BE be joined. Since AB is perpendicular to FG, ABD, ABE are right angles ^a, and because CD is parallel to AB, CDB is a right angle; it is to be proved that CDE is also a right angle.

The triangles ABD, BDE have the side AB = DE, BD common, and the angles at B, D right; therefore the base AD = BE ^b. Hence the triangles ADE, ABE are mutually equilateral; therefore the angle ADE = ABE ^c, and consequently a right angle. Thus, ED being perpendicular to both the lines AD, DB, is perpendicular to their plane BC ^d; therefore CDE is a right angle ^a, and consequently CD being perpendicular to both the lines BD, DE, is perpendicular to their plane FG ^d.

Conversely, If CD be perpendicular to FG, it is parallel to AB. For it may be proved in the same manner that ADE is a right angle. But BDE is a right angle by construction, and CDE a right angle by hypothesis. Therefore, since ED is perpendicular to the three lines BD, AD, CD, at the same point, these lines are all in one plane ^e. Consequently AB, CD are in the same plane; and since they are also perpendicular to the same line BD, they are parallel.

P R O P O.

PROPOSITION IV.

Straight lines parallel to the same are parallel to one another, though they be not all in the same plane.

Let the straight lines AB, CD, be each of them parallel to the straight line EF, but not both in the same plane with EF, AB is parallel to CD. Fig. 125.

For, from any point H, in the line EF, let GH be drawn perpendicular to it in the plane AF, and HK perpendicular to it in the plane FC. Then EF is perpendicular to the plane of the lines GH, HK^a; ^a 2. 6. therefore AB, CD, which are parallel to EF, are perpendicular to the same plane GHK^b, and consequently they are parallel to one another^b. ^b 3. 6.

PROPOSITION V.

To draw a straight line perpendicular to a given plane from a given point without it.

Let AB be the given plane, and C the given point Fig. 126. without it, from which it is required to draw a straight line perpendicular to AB. Let any straight line DE be drawn in the plane AB. In the plane CDE let CF be drawn perpendicular to DE, in the plane

plane AB , let FG be also perpendicular to DE , and in the plane CFG , CG perpendicular to FG : Then CG shall be perpendicular to AB .

For, let GH be parallel to DE . Then, because DE is perpendicular to both the lines CF , FG , it is perpendicular to their plane^a ; therefore GH , which is parallel to DE , is perpendicular to the same plane CFG ^b. Thus, CGH is a right angle ; but CGF is a right angle by construction ; consequently CG is perpendicular to the plane of the lines FG , GH ^c, that is, to AB .

COR. Hence a straight line may be drawn perpendicular to a given plane, from a given point in the same. For it may be drawn parallel to a perpendicular drawn from any point assumed without the plane.

PROPOSITION VI.

If two planes intersect each other, and from points in the line of common section straight lines be drawn perpendicular to it in each plane, the angles formed by these perpendiculars shall be equal.

Fig. 127. Let BE be the line of common section of the two planes AE , BF . Let AB , DE be perpendicular to BE in the former, and BC , EF perpendicular to it in the latter ; the angle ABC shall be equal to DEF .

For,

For, let AB, BC, DE, EF, be all equal, and AC, CF, FD, DA, be joined. Because AB, DE are both perpendicular to BE, and in the same plane, they are parallel. But AB, DE are also equal. Therefore AD is equal and parallel to BE^a. In like manner, ^a 23. 1. CF is equal and parallel to BE. Consequently AD is equal and parallel to CF^b, and AC equal and parallel to DF^c. ^b 4. 6. Thus, the triangles ABC, DEF are mutually equilateral; therefore the angle ABC is equal to the angle DEF^c. ^c 6. 1.

PROPOSITION VII.

If a straight line be perpendicular to a plane, any plane touching it will be perpendicular to the same.

Let the straight line AB be perpendicular to the plane CD, then any plane CE, touching AB, will be perpendicular to CD. Fig. 128.

For, let CBG be their line of common section, and from any point G, in CG, let GE be perpendicular to it in the plane CE. Also, let BF, GH be perpendicular to CG, in the plane CD. Then, because AB is perpendicular to CD, ABF is a right angle; therefore the angle EGH, contained by the perpendiculars at G, is also a right angle^a. But EGC is a right angle by construction. Wherefore EG is perpendicular

- ^b 2. 6. cular to the plane CD ^b. In like manner, HG is perpendicular to the plane CE. Consequently the planes
^c def. 3. 6. CE, CD are perpendicular to each other ^c.

COR. A straight line, drawn at right angles to one of two perpendicular planes, through any point in the other, meets the former in the line of common section, and lies wholly in the latter.

PROPOSITION VIII.

If two planes cutting each other be perpendicular to the same plane, their line of common section is also perpendicular to it.

Fig. 129. Let the two planes AB, CD, cutting each other in the line EF, be both perpendicular to the plane GH, then EF is perpendicular to the same plane.

For the perpendicular to GH, at the point F, where
^c c. 7. 6. EF meets it, is wholly in each of the planes AB, CD ^c, and consequently is the same with their line of common section.

PROPOSITION IX.

If two planes, or a straight line and a plane, be each of them perpendicular to the same straight line, they are parallel to one another.

Let

Let the two planes AB, CD be each of them perpendicular to the same straight line EF, at the points E, F; AB, CD are parallel. Fig. 130.

For, if not, let them meet in the line GH, and let EG, GF be joined. Then, because EF is perpendicular to AB, CD, it is perpendicular to the lines EG, FG^a, which are in their planes. Thus, EGF is a ^a def. 1. 6. triangle, having two of its angles right, which is impossible. Therefore the plane CD does not meet AB. But any straight line, as FL, perpendicular to EF at the point F, is in the plane CD^b. Consequently the ^b 6. c. 2. 6. straight line FL is also parallel to AB.

PROPOSITION X.

If two parallel planes be cut by a third plane, the lines of common section are parallel.

Let AB, CD, be two parallel planes, cut by a third plane EG, the lines of common section EF, GH are parallel. Fig. 131.

For, if not, let them be produced, and meet in the point K. Then, because the straight line FEK every where touches the plane AB, and the straight line GHK the plane CD, the point K is in each of the planes^a. Thus, the planes AB, CD meet at the ^a def. 5. 1. point K, which is contrary to hypothesis.

COR.

COR. 1. If two planes, cutting each other, be parallel to other two planes also cutting each other, the lines of common section are parallel.

COR. 2. If a plane be parallel to a straight line, its line of common section, with any plane touching that line, will also be parallel to the same.

COR. 3. If two straight lines be parallel, each of them is parallel to any plane touching the other, except that which is common to both.

COR. 4. A straight line, perpendicular to one of two parallel planes, is perpendicular to the other, and the planes are everywhere equidistant: Also any plane perpendicular to the one is perpendicular to the other.

COR. 5. If equal perpendiculars be erected from points in the same plane, their other extremes are also in one plane, parallel to the former.

COR. 6. Planes parallel to the same plane are parallel to one another.

COR. 7. If a straight line be parallel to a plane, any straight line which meets them both, and is perpendicular to the latter, will also be perpendicular to the former.

PROPO.

PROPOSITION XI.

If the sides of two angles which are not in the same plane be parallel, each to each, the angles are equal, and their planes parallel.

Let ABC , DEF be two angles not in the same plane, which have their sides parallel, each to each, AB to DE , and BC to EF , the angles ABC , DEF shall be equal, and their planes parallel. Fig. 132.

For, from the point B , let BG be drawn perpendicular to the plane DEF , and from G , GH parallel to DE , and GK to EF . Then, because BG is perpendicular to the plane DEF , BGH is a right angle. But, since AB , GH are both parallel to DE , they are parallel to one another. Therefore GBA is also a right angle. Thus, BG is perpendicular to the lines BA , GH . In like manner, it is perpendicular to the lines BC , GK . Therefore BG is perpendicular to the planes ABC , DEF , and consequently these planes are parallel. 5. 6.

Again, because AB , GH are parallel, they are in the same plane, and, for the same reason, BC , GK are both in one plane. Thus, $ABGH$, and $CBGK$, are two planes intersecting each other in the line BG , to which lines AB , GH are perpendicular in the one plane, and BC , GK in the other; therefore the angle

ABC

- * 6. 6. ABC is equal to HGK^c . But HGK is equal to
 * 17. 1. DEF^d . Consequently the angle ABC is equal to
 DEF .

PROPOSITION XII.

If perpendiculars be drawn to a plane, from two points in any straight line inclined to it, they shall have the same ratio as the segments of the line which the plane intercepts from these points.

Fig. 133. From the points A, B , in the straight line AB , inclined to the plane CD , and meeting it in the point E , let the perpendiculars AG, BF be drawn to CD , $AG : BF :: AE : EB$.

- Becausc AG, BF are both perpendicular to CD ,
 * 3. 6. they are parallel ^a. Hence the straight lines AG, AB, BF , are all in one plane, and F, E, B are points in its line of common section with CD . But
 * 1.c.1. 6. the common section of two planes is a straight line ^b. Therefore F, E, B , are in the same straight line; and the right angled triangles AEG, BEF , being
 * 4. 5. equiangular, $AG : BF :: AE : EB^c$.

COR. 1. The straight line which joins the extremities of the perpendiculars, passes through the point where the inclined line intersects the plane; and is there divided into segments, having the same ratio as the perpendiculars.

COR.

- 6. 6. ABC is equal to HGK^c. But HGK is equal to
 • 17. 1. DEF^d. Consequently the angle ABC is equal to
 DEF.

PROPOSITION XII.

If perpendiculars be drawn to a plane, from two points in any straight line inclined to it, they shall have the same ratio as the segments of the line which the plane intercepts from these points.

Fig. 133. From the points A, B, in the straight line AB, inclined to the plane CD, and meeting it in the point E, let the perpendiculars AG, BF be drawn to CD, $AG : BF :: AE : EB$.

- Becausè AG, BF are both perpendicular to CD,
 • 3. 6. they are parallel^a. Hence the straight lines AG, AB, BF, are all in one plane, and F, E, B are points in its line of common section with CD. But
 • 1.c.1. 6. the common section of two planes is a straight line^b. Therefore F, E, B, are in the same straight line; and the right angled triangles AEG, BEF, being
 • 4. 5. equiangular, $AG : BF :: AE : EB$ ^c.

COR. 1. The straight line which joins the extremities of the perpendiculars, passes through the point where the inclined line intersects the plane; and is there divided into segments, having the same ratio as the perpendiculars.

COR.

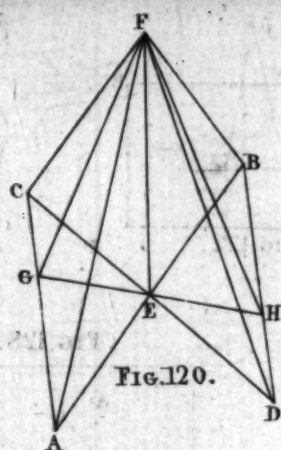


Fig. 120.

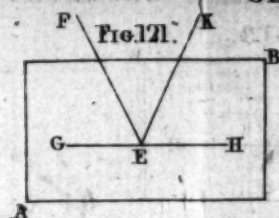


Fig. 121.

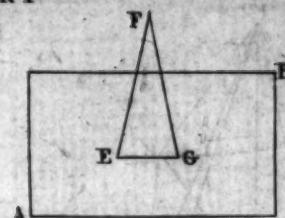


Fig. 122.

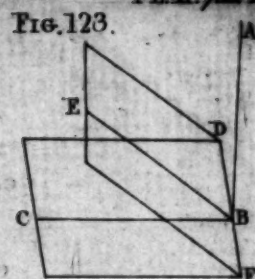


Fig. 123.

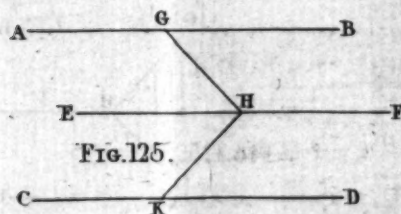


Fig. 125.

Fig. 128.

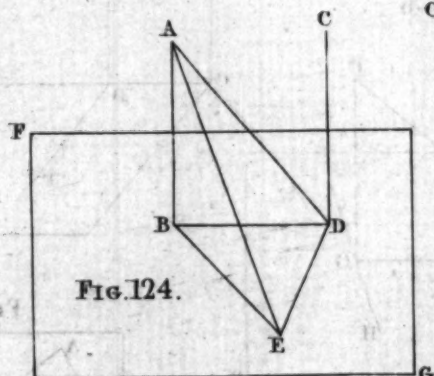
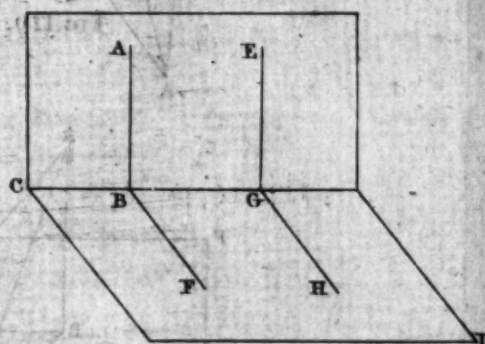


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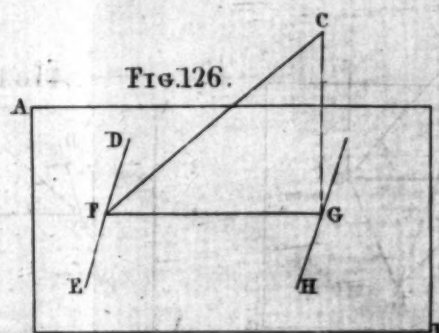


Fig. 126.

Fig. 129.

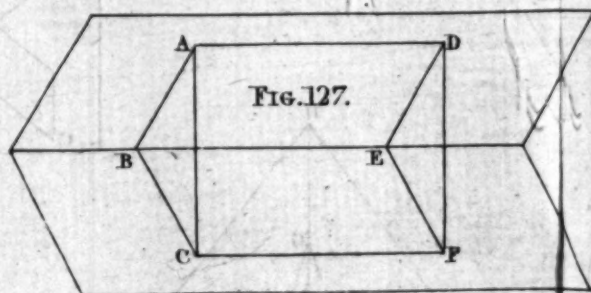
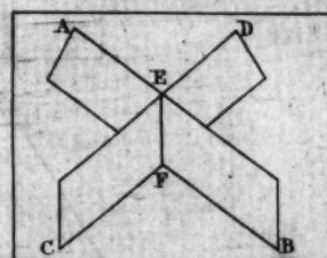


Fig. 127.

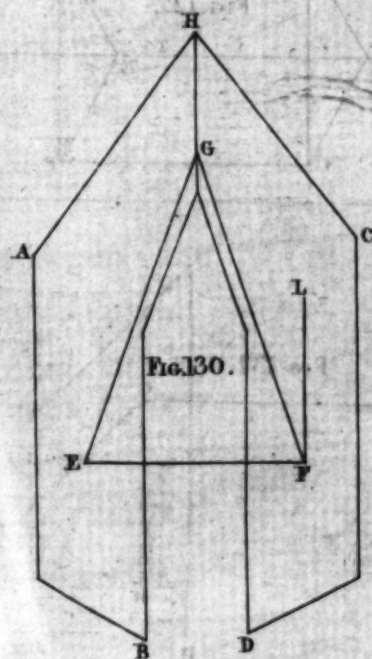


Fig. 130.

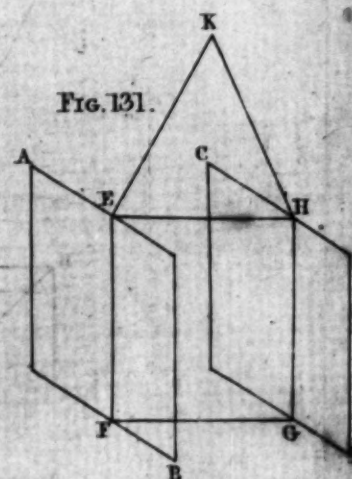


Fig. 131.

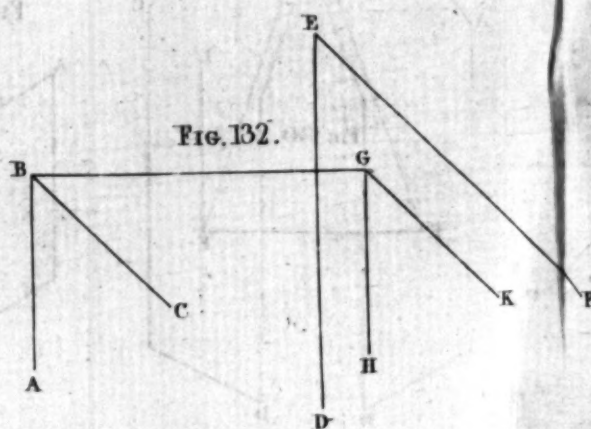


Fig. 132.

COR. 2. Straight lines, which meet three parallel planes, are cut proportionally.

PROPOSITION XIII.

If a solid angle be contained by three plane angles, any two of them are together greater than the third.

Let there be a solid angle at the point A, contained by the three plane angles BAC, CAD, DAB; any two of these are together greater than the third. Fig. 134.

If all the angles be equal, or if the two greater be equal, the proposition is evident. In any other case, let BAC be the greatest angle, and let BAE be cut off from it, equal to DAB. Through any point E, in AE, let the straight line BEC be drawn, in the plane of the angle BAC, to meet its sides in B, C. Let AD be equal to AE, and BD, DC joined. Because the triangles BAD, BAE have the side AD = AE, BA common, and the included angles equal, the base BD is equal to BE^a. But the sum of BD, DC is ^a 4. 1. greater than the sum of BE, EC^b. Therefore DC is ^b 20. 1. greater than EC; and, since the side DA is equal to AE, and AC common to the two triangles ACD, ACE, the angle CAD is greater than EAC^c. Con- ^c c 22. 1. sequently, the sum of the angles BAD, CAD is greater than the angle BAC.

PROPO.

P R O P O S I T I O N X I V .

All the plane angles which form any solid angle, are together less than four right angles.

Fig. 135.

Let there be a solid angle at the point A, contained by the plane angles BAC, CAD, DAE, EAB, these angles, taken together, are less than four right angles.

For, through any point B, in AB, let a plane be extended to meet the sides of the solid angle in the lines of common section BC, CD, DE, EB, thereby forming the pyramid BCDE — A : And from any point O, within the rectilineal figure BCDE, which is the base of the pyramid, let the straight lines OB, OC, OD, OE, be drawn to the angular points of the figure, which will thereby be divided into as many triangles as the pyramid has sides. Then, because each of the solid angles at the base of the pyramid is contained by three plane angles, any two of them are together greater than the third *. Thus, ABE, ABC are together greater than EBC. Therefore the angles at the bases of the triangles, which have their vertex at A, are together greater than the angles at the bases of the triangles which have their vertex at O. But all the angles of the former triangles are together equal to all the angles of the latter. Therefore

* 13. 6.

fore the remaining angles at A are together less than the remaining angles at O, that is, less than four right angles ^b.

^b 2. c. 11.
1.

PROPOSITION XV.

If two solid angles be each of them contained by three plane angles, and have these angles equal, each to each, and alike situated, the two solid angles are equal.

Let there be a solid angle at the point A, contained by the three plane angles BAC, CAD, DAB, and a solid angle at the point E, contained by the three plane angles FEG, GEH, HEF, equal to the former, each to each, and alike situated; these solid angles are equal. Fig. 136.

For, let the straight lines AB, AC, AD, EF, EG, EH, be all equal, and let their extremes be joined by the lines BC, CD, DB, FG, GH, HF: And thus there are formed two isosceles pyramids, BCD-A and FGH-E. Upon the bases BCD, FGH, let the perpendiculars AK, EL fall from the vertices A, E. Then, because the right angled triangles AKB, AKC, AKD have equal hypotenuses, and the side AK common, their other sides KB, KC, KD, are also equal^a:
Therefore K is the center of the circle that circumscribes the triangle BDC. In like manner, L is the
Y center

^a 2. c. 23.
1.

center of the circle that circumscribes the triangle FHG. But these triangles are equal in every respect^b.

^c 4. 1. For the sides BC, FG are equal^c, because they are the bases of equal and similar triangles BAC, FEG; and, for the same reason, CD is equal to GH, and DB to HF. If, therefore, the pyramid BCD-A be applied to the pyramid FGH-E, their bases BCD and FGH will coincide. Also, the point K will fall upon L, because, in the plane of the circle FHG, no other point than the center is equally distant from three points in its circumference^d; the perpendicular KA will be in the same straight line with LE, because there cannot be more than one perpendicular to a plane at the same point^e; and the point A will coincide with E, because the right angled triangles AKB, ELF having equal hypotenuses, and BK = FL, their other sides AK, EL are also equal^f. But the points B, D, C coincide with F, H, G, each with each; therefore the straight lines AB, AC, AD coincide with EF, EG, EH, and the plane angles BAC, CAD, DAB, with FEG, GEH, HEF, each with each. Consequently, the solid angles themselves coincide, and are equal.

COR. 1. If two solid angles, each contained by three plane angles, have their linear sides, or the planes that bound them, parallel each to each, the solid angles are equal^g.

^g 11. 6.
1. c. 10. 6.

COR.

COR. 2. If two solid angles, each contained by the same number of plane angles, have their linear or plane sides parallel, each to each, the solid angles are equal.

For each of the solid angles may be divided into solid angles, each contained by three plane angles, and the parts being equal and alike situated, the wholes are equal.

PROPOSITION XVI.

If two triangular prisms have the plane angles and linear sides, about two of their solid angles equal, each to each, and alike situated, the prisms are equal and similar.

Let ABC-DEF and GHK-LMN, be two triangular prisms, having the plane angles BAC, CAD, DAB, about the solid angle at A, equal to the plane angles HGK, KGL, LGH, about the solid angle at G, each to each, and the linear sides AB, AC, AD, equal to the linear sides GH, GK, GL, each to each; these prisms are equal and similar. Fig. 137.

For, since the solid angles A, G are each contained by three plane angles, which are equal each to each, these solid angles, being applied to each other, coincide*. Therefore the straight lines AB, AC, AD, coincide with the straight lines GH, GK, GL, each with each. But AB coinciding with GH, and the point D with L, the straight line DE must fall upon LM,

b 16. 1.

LM, because there cannot be more than one parallel to the same line drawn from the same point^b; so likewise the straight line BE falls upon HM: Therefore the point E coincides with M. In like manner, the point F coincides with N. Consequently, the rectilineal figures which bound the one prism coincide with, and are equal and similar to the rectilineal figures which bound the other, each to each; the solid angles of the one coincide with, and are equal to the solid angles of the other, each to each; and the solids themselves are equal and similar.

COR. 1. If two triangular pyramids have the plane angles, and linear sides, about two of their solid angles, equal each to each, and alike situated, the pyramids are equal and similar.

COR. 2. If two triangular prisms have the linear sides about two of their solid angles both equal and parallel, each to each, the prisms are equal and similar.

COR. 3. If two pyramids have the linear sides about their vertical angles both equal and parallel, each to each, the pyramids are equal and similar.

PROPOSITION XVII.

The opposite sides of a parallelepipedon are similar and equal parallelograms, and the diagonal plane divides it into two equal and similar prisms.

Let

Let ABCD-EFGH be a parallelopipedon, or solid, Fig. 133. bounded by six planes, of which the opposite ones are parallel, any two of its opposite sides, as ABCD and EFGH, are similar and equal parallelograms.

For, since the opposite planes AF, DG are parallel, and cut by a third plane BD, the lines of common section AB, DC are parallel^a. Also, because the opposite planes BG, AH are parallel, and cut by the plane BD, the lines of common section, AD, BC, are parallel. Thus, the figure ABCD is a parallelogram. In like manner, all the plane figures which bound the solid are parallelograms. Hence AD is equal to EH, and DC to HG^b, and the angle ADC, by reason of^b 1. 2. parallel lines, is equal to EHG^c; therefore the triangles ADC, EHG^c; consequently the parallelograms ABCD, EFGH^c are equal and similar.

Again, because AE, CG are each equal and parallel to DH, they are equal and parallel to each other; therefore the figure ACGE is a parallelogram^c, and divides the whole parallelopipedon AG into two triangular prisms ABC-EFG, and ACD-EGH. And these prisms are equal and similar^f, because the plane angles and linear sides about the solid angles at F and D are equal, each to each: For the plane angles EFG, ADC are equal to one another, as being each equal to EHG^c, and the linear sides FG, DA equal to one another, as being each equal to EH, and so of the others.

COR.

COR. 1. The opposite solid angles of a parallelopipedon are equal.

COR. 2. Every rectangular parallelopipedon is bounded by rectangles.

COR. 3. The ends of a prism are similar and equal figures, and their planes parallel.

COR. 4. Every parallelopipedon is a quadrangular prism, of which the ends are parallelograms; and conversely.

COR. 5. If two parallelopipedons have the plane angles, and linear sides, about two of their solid angles equal, each to each, and alike situated, or the linear sides, about two of their solid angles, both equal and parallel, each to each, the solids are equal and similar.

COR. 6. If, from the angular points of any rectilinear plane figure, there be drawn straight lines above its plane, all equal and parallel, and their extremes be joined, the figure formed thereby is a prism. If the rectilinear figure be a parallelogram, the prism is a parallelopipedon. If the rectilinear figure be a rectangle, and the straight lines be perpendicular to its plane, the prism is a rectangular parallelopipedon.

COR. 7. Hence the method of forming a parallelopipedon, of which one of the solid angles, and its linear sides, are given; also of forming a parallelopipedon, from a given triangular prism, as one of its halves.

P R O P O.

PROPOSITION XVIII.

Parallelopipedons upon the same base, and between the same parallel planes, are equal.

Let ABCD-EFGH, and ABCD-KLMN, be two parallelopipedons upon the same base ABCD, and between the same parallel planes AC, EM; the parallelopipedons AG, AM are equal. Because the straight lines EF, GH, KL, MN, are parallel to AB, CD, they are parallel to one another, and, being all in the same plane, NK, ML, will therefore meet both EF and GH^a; let them meet the former in O, P, and the latter in R, Q, and let AO, BP, CQ, DR, be joined. The figure ABCD-OPQR is a parallelopipedon: For, by hypothesis, the plane EQ is parallel to AC, the plane of the parallels DC-HQ is parallel to the plane of the parallels AB-EP, and the plane AD-NO parallel to BC-MP. Hence AEO-DHR and BFP-CGQ are two triangular prisms, which have the linear sides AE, AD, AO, about the solid angle at A, both equal and parallel to the linear sides BF, BC, BP, about the solid angle at B^b, each to each. Consequently, these prisms are equal^c, and each of them being taken away from the whole solid ABCD-EPQH, the remainders, the parallelopipedons AG, AQ are equal. In the same manner, the parallelopipedon AM may be proved equal

Fig. 139.

^a 2.c.16.1.

^b 17. 6.

^c 16. 6.

equal to AQ . Therefore the parallelopipedons AG and AM are equal to one another.

PROPOSITION XIX.

Parallelopipedons, upon equal bases, and between the same parallel planes, are equal.

Fig. 140.

CASE 1. When the bases have one side common, and lie between the same parallel lines. Let $ABCD$ - $EFGH$ and $ABKL$ - $OPQR$ be two parallelopipedons, between the parallel planes AK , EN , and upon the bases $ABCD$, $ABKL$, which have the side AB common, and lie between the parallel lines AB , DK ; these parallelopipedons AG , AQ are equal.

For, let the planes EAL , FBK meet the side HC of the former in the lines of common section LM , KN , and the side HF in the lines EM , FN . Then, since

• 17. 6. EA , AL are parallel to FB , BK , each to each^a, their

• 11. 6. planes are parallel^b, and the figure AN is a parallelopipedon. Hence ADL - EHM and BCK - FGN are two triangular prisms, which have the linear sides AD , AE , AL about the solid angle at A , both equal and parallel to the linear sides BC , BF , BK , about the solid angle at B , each to each. Consequently, these

• 16. 6. prisms are equal^c, and each of them being taken away from the whole solid $ABKD$ - $EFNH$, the remainders,

ders, the parallelopipedons AG, AN are equal. But AN is equal to AQ^d, because they are upon the same base, and between the same parallel planes. Therefore the parallelopipedons AG, AQ are equal to one another.

CASE II. When the bases are equiangular. Let Fig. 141. ABCD, CEFG be equal bases of two parallelopipedons, between the same parallel planes, and let them have the angles BCD, GCE equal. Let DC, CE be in the same straight line, then BC, CG are also in one straight line^e: Also, let their sides be produced to meet in the points H, K^f. Since the parallelograms AC, CF are equal, they are complements of the parallelogram AF, and the straight line HCK is its diagonal^g. Upon the base AF, let a parallelopipedon be erected of the same altitude with those upon AC, CF^h; and let it be cut by planes parallel to its sides, and touching the lines DCE, GCB; These planes divide the whole parallelopipedon into four other parallelopipedons upon the bases DG, AC, BE, CF. But the diagonal plane touching HK divides each of the parallelopipedons upon AF, DG, BE into two equal prismsⁱ. Therefore the prisms upon HGC, CEK, are together equal to the prisms upon HDC, CBK; and these being taken away from the equal prisms upon HFK, HAK, the remainders, the parallelopipedons erected upon AC, CF are equal. Consequently

^d 18. 6.

^e c. 12. 1.

^f 2. c. 16. 1.

^g c. 1. 2.

^h c. 6. 17. 6.

ⁱ 17. 6.

shutle has late and minute
Z
9 9 9 9

18. 6.

Fig. 142.

frequently any two parallelopipedons upon these bases, and between the same parallel planes, are equal *.

CASE III. When the bases are neither equiangular, nor have one side common. Let $ABCD$, $EFGH$ be the equal bases of two parallelopipedons, between the same parallel planes. At the point E , let the angle FEL be made equal to BAD , let GH meet EL in L , and let the parallelogram FL be completed. Because the parallelograms AC , FL are each equal to EG , they are equal to one another; and they are also equiangular; Therefore the parallelopipedon upon AC is equal to any parallelopipedon upon FL , between the same parallel planes. But the parallelopipedon upon EG is equal to the same. Therefore the parallelopipedons upon AC , EG , are equal to one another.

COR. 1. Parallelopipedons of equal bases and equal altitudes are equal.

COR. 2. Triangular prisms, upon equal bases, which are either both triangles, or both parallelograms, and of equal altitudes, are equal.

COR. 3. Two triangular prisms, upon a parallelogram and a triangle, as their bases, and of equal altitudes, are equal to one another, if the parallelogram be double the triangle.

COR. 4. A triangular prism, and consequently any other prism, upon one of its ends, as a base, is equal to a parallelopipedon of equal base and altitude.

P R O P Q.

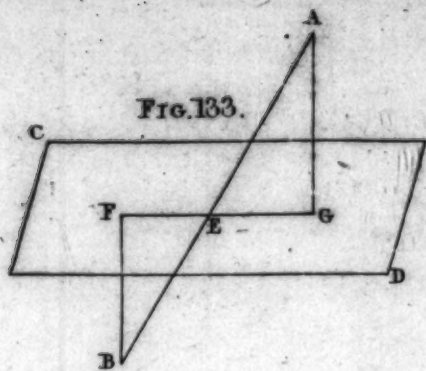


FIG. 133.

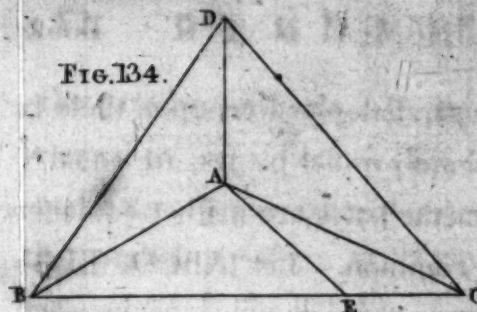


FIG. 134.

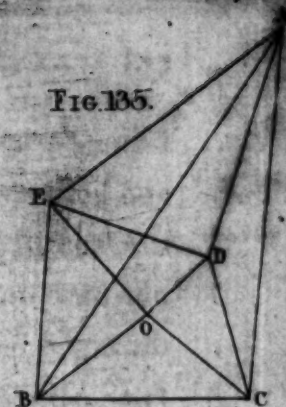


FIG. 135.

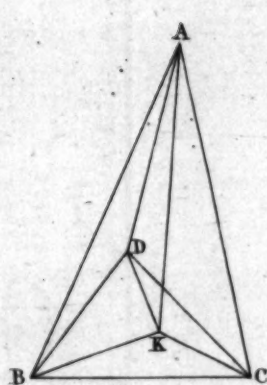


FIG. 136.

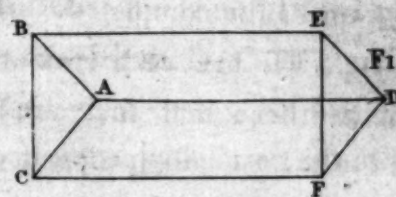
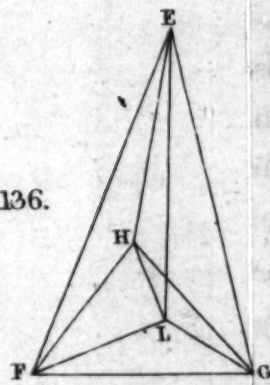


FIG. 137.

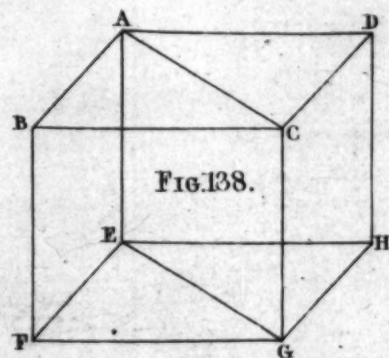
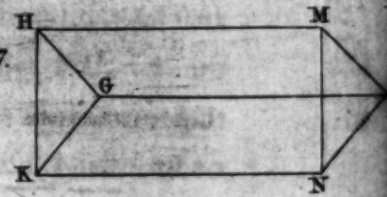


FIG. 138.

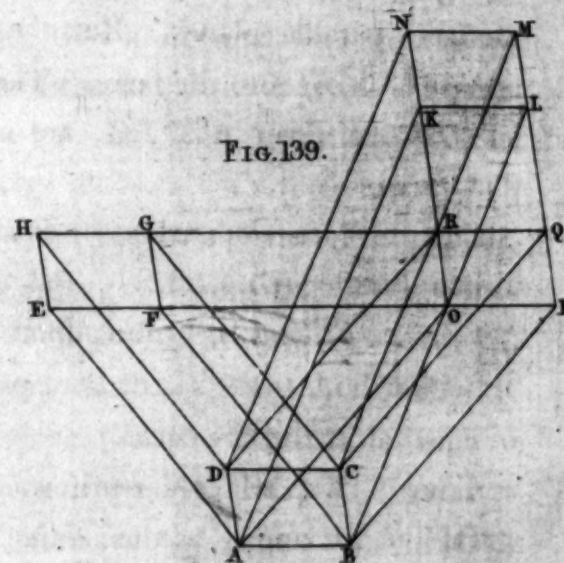


FIG. 139.

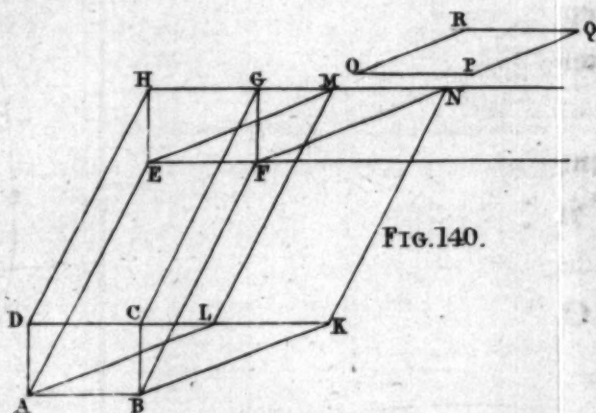


FIG. 140.

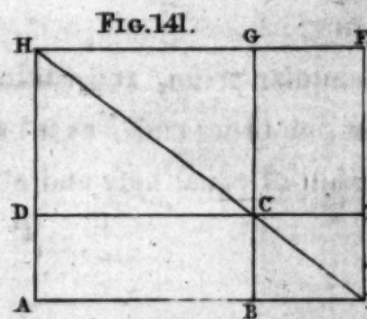


FIG. 141.

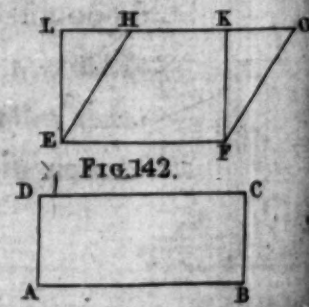


FIG. 142.

THE HISTORY OF THE

REIGN OF

CHARLES THE FIRST

BY

JOHN BURNET

OF THE UNIVERSITY OF OXFORD

IN TWO VOLUMES

VOLUME THE FIRST

THE HISTORY OF THE

REIGN OF

CHARLES THE FIRST

BY

JOHN BURNET

OF THE UNIVERSITY OF OXFORD

IN TWO VOLUMES

VOLUME THE FIRST

THE HISTORY OF THE

REIGN OF

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BY

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OF THE UNIVERSITY OF OXFORD

IN TWO VOLUMES

PROPOSITION XX.

Parallelopipedons of equal altitudes are to one another as their bases.

Let ABCD, EFGH be the bases of two parallelopipedons AK, EL, of equal altitudes, $AK : EL :: AC : EG$. Fig. 143.

For, let the base AB, of the parallelogram AC, be produced to M, so that BM be a fourth proportional to the altitudes of AC, EG, and the base EF, and let the parallelogram CM, and the parallelopipedon MK be completed. Let Bn be any part whatever of the line BM, let no be parallel to BC, and let the parallelopipedon MK be cut by the plane noqr parallel to BK. Then, because parallelopipedons of the same altitude, upon equal bases, are equal^a, the parallelopipedon MK will contain Bq as often as the parallelogram CM contains Bo, or the line BM contains Bn. Thus, Bq may be any part whatever of KM. But, for the same reason, the parallelopipedon AK contains Bq just as often as the parallelogram AC contains Bo. Consequently $AK : KM :: AC : CM$. But, from the construction, it is evident that the parallelogram EG is equal to CM^b; therefore the parallelopipedon EL is equal to KM^c, and $AK : EL :: AC : EG$. 19. 6.

^b 7. c. 8. 5.

^c 1. c. 19. 6.

COR.

COR. Prisms standing upon their ends, and of equal altitudes, are to one another as their bases.

P R O P O S I T I O N XXI.

Two parallelopipedons, which have a solid angle of the one equal to a solid angle of the other, are to one another in the ratio compounded of the ratios of the linear sides of the one to the linear sides of the other, each to each, about these solid angles.

Fig. 144. Let DE, KL be two parallelopipedons, of which the solid angle at B, in the former, is contained by plane angles ABC, ABE, EBC, respectively, equal to the plane angles FGH, FGL, LGH, containing the solid angle at G in the latter, and alike situated; then the ratio DE : KL is the same with that which is compounded of the ratios AB : FG, CB : HG, and EB : LG.

For, let AB, CB, EB, be produced to M, N, O, so that BM = FG, BN = GH, and BO = GL. With the lines EB, BM, and BN, let the parallelopipedon

^a 7. c. 17. EP be completed ^a, and with the lines OB, BM, and BN, the parallelopipedon OP. Then, because the ratio DE : OP is compounded of the two ratios DE : EP, and EP : OP, of which the former is equal to

^b 20. 6. AC : MN ^b, and the latter to EM : MO ^b, or EB : BO ^c;
^c 2. c. 1. 5. the

the ratio $DE : OP$ is the same with that which is compounded of $AC : MN$, and $EB : BO^c$, or of $AB : BM$, $CB : BN^c$, and $EB : BO$. But OP is equal to KL^r , because these parallelopipedons have the plane angles, and linear sides about their solid angles at B, G respectively equal and alike situated. Consequently the ratio $DE : KL$ is the same with that which is compounded of $AB : BM$, $CB : BN$, and $EB : BO$; or of $AB : FG$, $CB : HG$, and $EB : LG$.

COR. 1. Two rectangular parallelopipedons are to one another in the ratio compounded of the ratios of the linear sides of the one to the linear sides of the other, each to each: And any ratio compounded of three ratios (whose terms are straight lines,) is the same with the ratio of the rectangular parallelopipedons, under their homologous terms.

COR. 2. Two cubes, or, in general, two similar parallelopipedons, are to one another in the triplicate ratio of their homologous linear sides.

COR. 3. Similar parallelopipedons are to one another as the cubes of their homologous linear sides.

COR. 4. If four straight lines be in continued proportion, the first is to the fourth as the cube of the first to the cube of the second.

COR. 5. The rectangular parallelopipedons, under the corresponding terms of three analogies, are proportional.

COR.

COR. 6. If four straight lines be proportional, their cubes are also proportional; and conversely.

COR. 7. Rectangular parallelopipedons, and consequently any other parallelopipedons, are to one another in the ratio compounded of the ratios of their bases and altitudes.

COR. 8. Parallelopipedons of equal bases are to one another as their altitudes.

COR. 9. Parallelopipedons, whose bases and altitudes are reciprocally proportional, are equal; and conversely.

COR. 10. Prisms are to one another in the ratio compounded of the ratios of their bases and altitudes. Therefore the 8th and 9th Cor. may be applied to prisms.

PROPOSITION XXII.

Every triangular pyramid may be divided into two equal prisms, which are together greater than half the whole pyramid, and two equal pyramids, which are similar to the whole, and to one another.

Fig. 145.

Let BCD-A be any triangular pyramid. Let its linear sides AB, AC, AD, be bisected in E, F, G, and the points of section joined; and let the sides of the base CB, BD, DC, be bisected in H, K, L, and the

the points of section joined : Also, let EH, EK, and LG, be drawn. Because the two sides AB, AC of the triangle ABC, are bisected in E, F, the straight line EF is equal and parallel to CH or HB, half the remaining side BC^a; and so of the other straight lines that join the points of section. Hence EC, CG, and consequently EL, are parallelograms, and the planes EFG, BCD parallel. Therefore the solid EFG-HCL, upon the triangular base HCL, is a prism. And the solid EHK-GLD is a triangular prism of the same altitude, upon the parallelogram HD, which is double the triangle HCL. Consequently these prisms are equal^b. The two remaining solids EFG-A, and BHK-E, are triangular pyramids. They are equal and similar to one another, and also similar to the whole, because the triangles which bound the one are equal and similar to the triangles which bound the other, and also similar to those which bound the whole, each to each, and alike situated. But either of these pyramids is, for the same reason, equal to the pyramid KLD-G, and therefore less than either of the two prisms. Consequently, the two prisms EFG-HCL, and EHK-GLD, are together greater than the two pyramids EFG-A, and BHK-E, and greater than half the whole pyramid BCD-A.

COR. By taking from the whole pyramid the two equal prisms, and from each of the remaining pyramids two equal prisms, formed in like manner, there will

^a 2. 5.

^b 3. c. 19.
6.

will at length remain a magnitude less than any proposed.

PROPOSITION XXIII.

Pyramids of equal altitudes are to one another as their bases.

Fig. 146.

CASE I. Let BCD-A, and NOP-M, be two triangular pyramids of equal altitudes, $BCD-A : NOP-M :: BCD : NOP$.

For, since the prisms EFG-HCL and QRS-TOX have equal altitudes, (each half the altitude of its pyramid^a), they are to one another as their bases^b, $EFG-HCL : QRS-TOX :: HCL : TOX$. But $HCL : TOX :: 4HCL : 4TOX :: BCD : NOP$. Therefore the ratio of the two prisms, in the pyramid BCD-A, to the two prisms in the pyramid NOP-M, is the same with that of the bases BCD, NOP. In like manner, the ratio of the two prisms in the pyramid EFG-A, to the two prisms in the pyramid QRS-M, is the same with that of (EFG, QRS, or HCL, TOX, or) BCD, NOP. Consequently, the sum of all the prisms taken from the pyramid BCD-A, after the same manner, by any number of divisions, is to the sum of all the prisms taken from the pyramid NOP-M, by the same number of divisions, as the base BCD is to the

^a 12. 6.

^b c. 20. 6.

the base NOP. And, since the remainder of either may be less than any magnitude proposed, the pyramids themselves are in the same ratio^c.

^c 18. 4.

Fig. 147.

CASE II. Let ABCDE-F and GHKL-M be two pyramids of equal altitudes, having any rectilineal figures for their bases, $ABCDE-F : GHKL-M :: ABCDE : GHKL$.

For, let the base ABCDE be divided into triangles by the diagonals AC, AD, then the whole pyramid will be divided into triangular pyramids, by planes touching these lines AC, AD, and passing through the vertex F: Also, let NOP-Q be any triangular pyramid of equal altitude.

Then, since the pyr. $ABC-F : NOP-Q :: ABC : NOP$

And . . . $ACD-F : NOP-Q :: ACD : NOP$

And . . . $ADE-F : NOP-Q :: ADE : NOP$

The whole pyr. $ABCDE-F : NOP-Q :: ABCDE : NOP$ ^d 15. 4.

In like manner $GHKL-M : NOP-Q :: GHKL : NOP$.

Hence, the terms of the last analogy being inverted, it follows, by equality, that $ABCDE-F : GHKL-M :: ABCDE : GHKL$.

PROPOSITION XXIV.

Every triangular prism may be divided into three equal triangular pyramids.

A a

Let

Fig. 148.

Let $ABC-DEF$ be a triangular prism, and let the diagonals BD , BF , of two of its sides, be drawn from the same point B ; the plane DBF cuts off a triangular pyramid $DEF-B$, and there remains a quadrangular pyramid $DACF-B$, which may be divided into two triangular pyramids $DAC-B$, and $DFC-B$, by a plane passing through its vertex B , and touching DC , the diagonal of its base. Thus, by the two planes DBF , DBC , the whole prism is divided into three triangular pyramids, having a common vertex at B , $DEF-B$, $DAC-B$, and $DFC-B$. Of these the second and third are equal^{*}, because they are upon equal bases, and have the same altitude. But the first and third are also equal, for they are the same with $BEF-D$ and $BCF-D$, which are upon equal bases, and have the same altitude. Therefore the prism is divided into three equal triangular pyramids.

* 23. 6.

COR. 1. Every pyramid is the third part of a prism of the same (or equal) base and altitude.

COR. 2. Pyramids are to one another in the ratio compounded of the ratios of their bases and altitudes.

COR. 3. Similar pyramids or prisms are to one another in the triplicate ratio of their altitudes, or their homologous linear sides.

COR. 4. Similar pyramids, or prisms, are to one another as the cubes of their homologous sides.

COR.

COR. 5. The rustum of a triangular pyramid may be divided, in the same manner, into three triangular pyramids, which are in continued proportion. For $DAC-B : DFC-B = (DAC : DFC = AC : DF = BC : EF = BCF : BEF =) BCF-D : BEF-D$.

PROPOSITION XXV.

A cone, and a pyramid, of equal altitudes, are to one another as their bases.

Let BD-H be a cone, and EF-G a pyramid of equal altitudes, BD-H : EF-G :: BD : EF. Fig. 149.

For, in the base of the cone, let the square ABCD be inscribed; and let the regular polygon KL be formed by bisecting the arches which the sides of the square subtend. The difference between the square pyramid ABCD-H and the cone BD-H is the sum of the conic segments AHB, BHC, CHD, DHA, cut off by the sides of the pyramid: But the triangular pyramid AKB-H is greater than half the conic segment AHB, in which it is inscribed, because it is equal to half the triangular pyramid BHM, that circumscribes AHB. If, therefore, the pyramid KL-H be taken from the cone BD-H, there will be taken away more than half that difference. In like manner, by taking from the same cone the inscribed pyramid of double the

23. 6.

the number of sides, there will be taken away more than half the remainder ; and so on. Therefore a pyramid may be taken from the cone that shall leave a remainder less than any magnitude proposed. And, since pyramids of equal altitudes are to one another as their bases, of the four magnitudes BD-H, EF-G, BD, EF, there may be taken from the first and third, magnitudes proportional to the second and fourth, so that the remainder of either shall be less than any proposed magnitude. Consequently these four magnitudes are proportional^b.

^b c. 18. 4.

COR. 1. It may be demonstrated, in like manner, that a cylinder, and a prism, of equal altitudes, are to one another as their bases.

COR. 2. A cone is equal to a pyramid, and a cylinder equal to a prism, of equal base and altitude.

COR. 3. A cone is the third part of a cylinder of the same (or equal) base and altitude.

COR. 4. Cones or cylinders are to one another, in the ratio compounded of the ratios of their bases and altitudes.

COR. 5. Similar cones or cylinders are to one another in the triplicate ratio of their axes, or of the diameters of their bases, or in the simple ratio of the cubes of these dimensions.

C O N I C

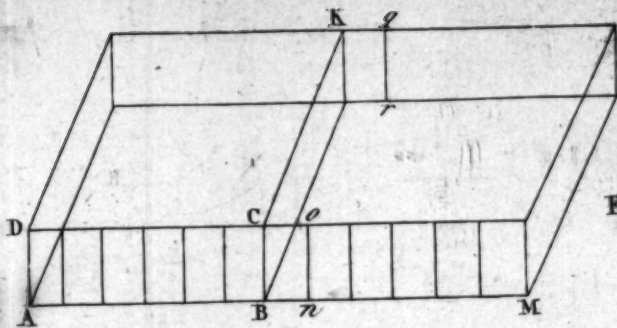


FIG. 143.

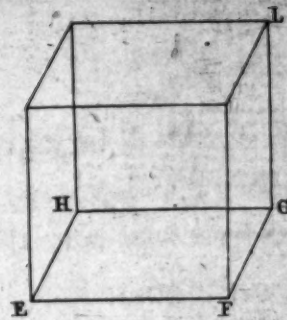


FIG. 144.

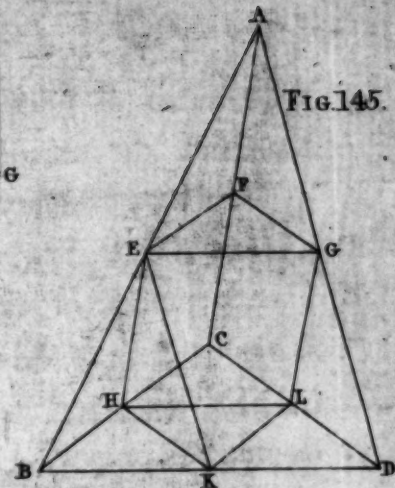
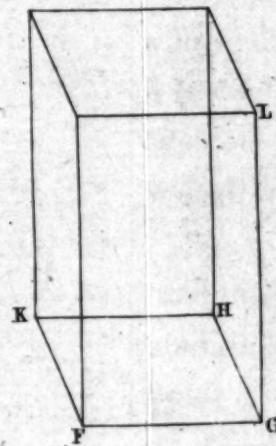
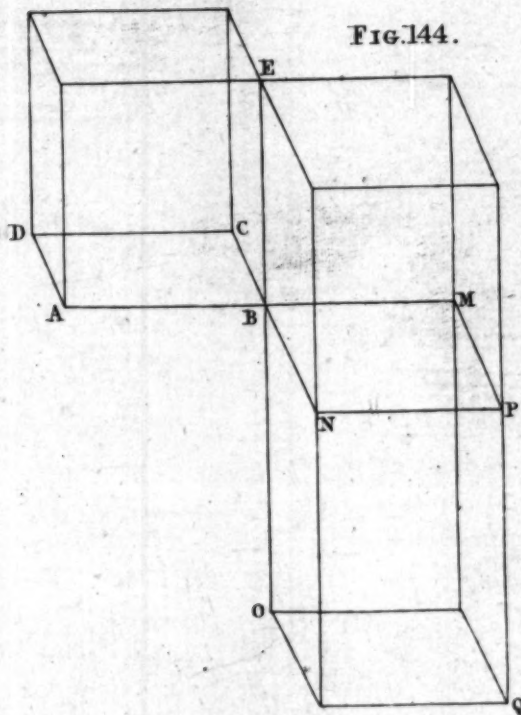


FIG. 145.

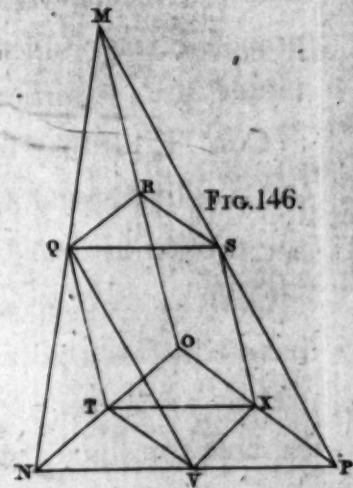


FIG. 146.

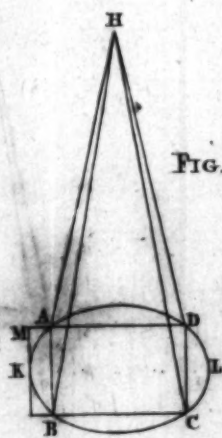


FIG. 149.

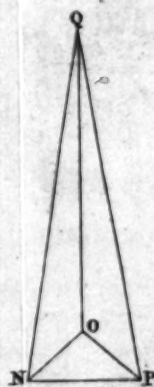
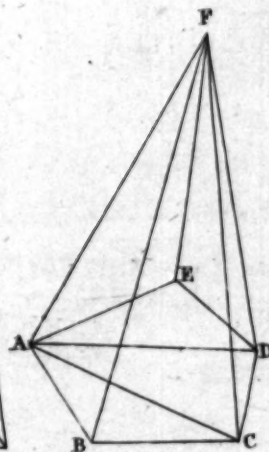


FIG. 147.

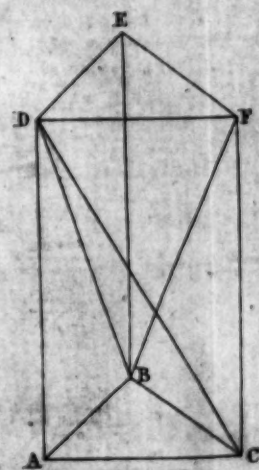
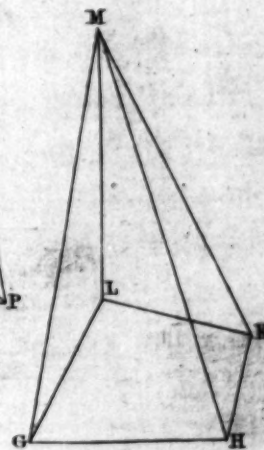


FIG. 148.

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CONIC SECTIONS.

B O O K I.

P A R A B O L A.

D E F I N I T I O N S.

1. **L**ET there be a straight line, and a point without it, given in position, and let a point move always equally distant from both, it will describe a curve, which is called a Parabola.

2. The given line is named the Directrix, and the given point the focus.

3. The vertex of the parabola is the middle of the perpendicular, which falls upon the directrix from the focus: And the axis, or principal diameter, is that part of the perpendicular produced indefinitely which falls within the curve.

4. Any

4. Any straight line, drawn from a point in the curve, parallel to the axis, and in the same direction, is called a diameter, and the point in the curve its vertex.

5. An ordinate to a diameter is a straight line, terminated on both sides by the curve, and bisected by that diameter: The part of the diameter, which it cuts off, is called an absciss.

6. The parameter of a diameter is four times the distance of its vertex from the directrix.

7. A tangent is a straight line, which meets the curve in one point only, and every where else falls without it.

8. A subtangent is that part of a diameter which is intercepted by a tangent, and an ordinate, from the point of contact.

PROPOSITION I.

The distance of a point from the focus is greater than its distance from the directrix, if the point be without the parabola, but less if it be within.

Fig. 1.

Let GVN be a parabola, whose directrix is AB, vertex V, and focus F; and let P be a point without the curve, that is, on the same side of the curve with the directrix: Then, if PF be joined, and PQ be drawn

drawn perpendicular to AB, PF will be greater than PQ.

For, as PF necessarily cuts the curve, let G be the point of section, GD perpendicular to AB, and P, D joined. Then, because $GF = GD^a$, $PF = PG + GD$. ^a def. 1. Hence PF is greater than PD ^b, and consequently still ^b 20. 1. G greater than PQ ^c. ^c 3. c 19. 1.

Again, let O be a point within the curve. The perpendicular OD, upon AB, necessarily cuts the curve; let G be the point of section, and let GF, FO be joined: Then OD is equal to the sum of OG, GF ^a, and therefore greater than OF ^b.

COR. 1. A point is without, in, or within the curve, according as its distance from the focus is greater, equal, or less, than its distance from the directrix.

COR. 2. A perpendicular to the axis, at its vertex, is a tangent to the parabola.

PROPOSITION II.

Every straight line perpendicular to the directrix meets the parabola, and every diameter falls wholly within it.

Let DG be perpendicular to the directrix, at any point of it, then shall DG meet the curve. Fig. 1.

For,

For, DF being joined, let the angle DFG be made
 * 2. c. 16. 1. equal to GDF, and let FG meet DG^a, which is parallel to FC, in G. The triangle DGF, having the angles at D and F equal, will also have the sides GD, GF equal; and therefore G is a point in the curve^b.

Again, the diameter GE falls wholly within the curve^b.

For, if any point O be assumed in it, it is evident that OD is greater than OF.

COR. Hence, the two legs of the curve continually diverge from the axis.

PROPOSITION III.

A straight line bisecting the angle formed by two lines drawn from the same point in the curve, the one to the focus, and the other perpendicular to the directrix, is a tangent to the parabola in that point.

Fig. 2. The straight line GE, bisecting the angle DGF, is a tangent to the parabola in G.

For, let H be any other point in GE, from which let there be drawn HF and HD, also HA perpendicular to AB. Then, because GE bisects the vertical angle of the isosceles triangle GDF, it will also bisect the base DF at right angles. Hence the triangles HED, HEF are equal in every respect^a. Thus, HF
 † 4. 1. is equal to HD, and therefore greater than HA^b.
 † 3. c. 19. 1. Consequently, the point H is without the curve^c.

COR.

COR. 1. Hence the method of drawing a tangent from any point in the curve.

COR. 2. If a straight line be drawn from the focus, to any point in the directrix, the perpendicular which bisects it will touch the parabola; also, every perpendicular to it, which cuts the curve, will be nearer to the focus than to the point in the directrix.

COR. 3. The parabola is concave towards the axis.

PROPOSITION IV.

Every straight line, drawn through the focus of a parabola, except the axis, meets the curve in two points.

Let FQ be a line passing through the focus, FD Fig. 3. perpendicular to it, DA, DE each equal to DF; AG, EH parallel to CF, and intersected by FQ, in G, H; * 2. c. 16. 1. and let the points AF be joined. Then, because DA is equal to DF, the angles DAF, DFA are equal; and these being taken from the right angles DAG, DFG, the remainders, the angles GAF, GFA are equal: Whence the sides GA, GF are also equal, and therefore G a point in the curve^b. In the same man-^b 1. c. 1. ner, it may be shewn that H is a point in the curve.

COR. A straight line, making an indefinitely small angle with the axis, being produced, meets the curve; hence the rate of divergency must be very small.

PROPOSITION V.

If, from any point in a parabola, a straight line be drawn, not parallel to a diameter, nor bisecting the angle formed by two lines drawn from that point, the one to the focus, and the other perpendicular to the directrix, it will meet the curve in one other point, and not in more than one.

Fig. 4.

Let H be a point in the curve, and HG a line not parallel to CF , nor bisecting the angle EHF ; then HG shall meet the curve in another point.

For, let FL be perpendicular to GH , and it will meet AB , since GH is not parallel to CF ; also, the point D , in which it meets AB , will be different from E ; since, if D, E were the same, it would follow that HG bisects the angle EHF , contrary to the hypothesis.

Let $DA = DE$, and AG parallel to EH , meet HG in G , G will be a point in the curve. About the center H , with the radius HE or HF , let a circle be described intersecting FD in K , and let another circle be described through the three points A, K, F . Then, because AB touches the circle EKF in E ^a, the rectangle $FDK = DE$ ^a^b $= DA$ ^a; hence DA is a tangent to the circle AKF ^c, and therefore AG passes through its center^d; but HL , which bisects the chord

FK ,

^a 11. 3.^b 1. c. 17. 3.^c 2. c. ib.^d 2. c. 11. 3.

FK, at right angles^e. also passes through its center^e; ^e 1. c. 2. 3. consequently G is the center of the circle AKF.

Whence $GA = GF$, and G a point in the parabola^r. ^r 1. c. 1.

If GH were to meet the curve in another point, that point would be the center of a circle passing through F, K, and touching the line AB in a point different from A, E, which is evidently impossible.

COR. 1. If a straight line be drawn through any point within the parabola, not parallel to a diameter, it will meet the curve in two points.

COR. 2. At the same point, in the curve, there cannot be more than one tangent.

COR. 3. A tangent bisects the angle formed by a straight line drawn to the focus, and another perpendicular to the directrix from the point of contact; it also bisects, and is perpendicular to the line that subtends that angle.

COR. 4. If a straight line from the focus be perpendicular to a chord, it will bisect that part of the directrix which is intercepted by perpendiculars falling upon it from the extremities of the chord; and conversely. If FL be perpendicular to GH, it will bisect AE. For the circle EKF being described about the center H, $FL = LK$, hence $GK = GF$, but $GF = GA$; therefore G the center of the circle AKF, and $DA^2 = FDK = DE^2$.

PROPOSITION VI.

A straight line terminated by the parabola, and parallel to a tangent, is an ordinate to the diameter that passes through the point of contact.

Fig. 5.

The chord GH, parallel to the tangent MQ, will be bisected by the diameter MN.

- For, let GA, HE be perpendicular to AB, and let FD meet MQ, GH in Q, L. Then, because FD is
^a 3. c. 5. perpendicular to MQ^a, it is also perpendicular to
^b 4. c. 5. GH, and therefore bisects AE in D^b. Whence, since
^c 1. c. 2. 5. AG, DN, EH, are parallel, GH is bisected in N^c.

COR. 1. An ordinate to any diameter is parallel to the tangent at the vertex of that diameter. For, if $GN = NH$, then $AD = DE$ ^c; therefore DF perpendicular to GH^b, and GH parallel to MQ.

COR. 2. The straight line which bisects two parallel chords is a diameter.

PROPOSITION VII.

The square of a semi-ordinate, to any diameter is equal to the rectangle under the parameter of that diameter, and the corresponding absciss.

CASE

CASE I. Let GNH be an ordinate to the axis; Fig. 6.
then $GN^2 = 4CV \times VN$.

For, $GN^2 = GF^2 - FN^2$: But $GF = GA = CN$; ^{a c. 9. 2.}
therefore $GN^2 = CN^2 - FN^2 = CN + FN \times CN - FN$ ^{b 8. 2.}
 $= 2VN \times CF$, and because $2VN : VN :: 4CV : CF$,
 $2VN \times CF = 4CV \times VN$. Hence $GN^2 = 4CV \times VN$.

CASE II. Let GNH be an ordinate to any other Fig. 5.
diameter MK: Then $GN^2 = 4DM \times MN$.

For, let the tangent MQ, the perpendiculars GA, GK, HE, and the lines DH, HF, FD, be drawn.
Then, because the right angled triangles FCD, DLN are equiangular, and MQ, LN parallel, $DF : FC :: DN : DL :: MN : LQ$, therefore $DF \times LQ = FC \times MN$, ^{c 3. c. 5. 5.}
and $2DF \times LQ (= 2FC \times MN) = 4CV \times MN$. But $2DF \times LQ = DH^2 - HF^2 = DE^2 =$ ^{d 11. 2.}
 $DA^2 = GK^2$. Consequently, $GK^2 = 4CV \times MN$.

Again, from the similar triangles DFC, DQM, we have $CF : FD :: DQ : DM$, therefore $CV : DQ ::$ ^{e 1. 4.}
 $DQ : DM$, and $CV : DM :: DQ^2 : DM^2$. But, ^{f 3 c. 10. 5.}
from the similar triangles GKN, DQM, $GK^2 : GN^2 :: DQ^2 : DM^2$. Whence $GK^2 : GN^2 :: CV : DM$ ^{g 5. c. 9. 5.}
 $:: 4CV \times MN : 4DM \times MN$. But $GK^2 = 4CV \times MN$, consequently $GN^2 = 4DM \times MN$.

COR. I. The square of a perpendicular, upon any diameter, from a point in the curve, is equal to the rectangle under the parameter of the axis, and the absciss corresponding to the ordinate from the same point.

COR.

COR. 2. If there be two diameters, and from the vertex of each a semi-ordinate be applied to the other, the abscisses will be equal.

COR. 3. The square of that part of a tangent, between the point of contact and any diameter, is equal to the rectangle under the external segment of that diameter, and the parameter of the diameter which passes through the point of contact.

COR. 4. The squares of ordinates, or semi-ordinates, to any diameter, are to one another as their corresponding abscisses; and the squares of perpendiculars, from the same points, are in the same ratio.

COR. 5. If, from points in the curve, two tangents be drawn to meet, their squares will be to each other as the parameters of the diameters passing through the points of contact.

COR. 6. If the squares of parallel lines, drawn from certain points, to meet a line given in position, be to one another as the parts they cut off towards one extremity, these points will be in the curve of a parabola, which has the given line for a diameter.

PROPOSITION VIII.

A subtangent, upon any diameter, is bisected in the vertex of that diameter.

Let the tangent GM meet any diameter VN in M , Fig. 6. and let GN be an ordinate to it, from the point of contact, the subtangent MN is bisected in V .

For, let the diameter GL , and its semi-ordinate VL be drawn, then is the absciss $GL = VN^a$; but, ^a 2. c. 7. since LM is a parallelogram ^b. $GL = MV$, therefore ^b 1. c. 6. $MV = VN$.

COR. 1. That part of the axis, between the focus and any tangent, is $\frac{1}{4}$ of the parameter of the diameter passing through the point of contact.

COR. 2. If $MV = VN$, and GN a semi-ordinate, then GM is a tangent, or, if GM be a tangent, GN is a semi-ordinate.

PROPOSITION IX.

That ordinate of a diameter which passes through the focus is equal to its parameter.

Let GE be any diameter, and RE the semi-ordinate Fig. 6. to it, which passes through the focus; then $2RE = 4GD$.

For,

^a 7.^b 1. c. 8.

For $RE^2 = 4GD \times GE^a$; but $GE = FM = GD^b$:
 Therefore $RE^2 = 4GD^2$, $RE = 2GD$, and $2RE = 4GD$.

COR. If an ordinate to any diameter pass through the focus, the absciss will be equal to the distance of the vertex from the focus.

PROPOSITION X.

If, from any point in the parabola, a parallel to a diameter be drawn to meet an ordinate to the same, the rectangle under the parameter of the diameter and the parallel will be equal to the rectangle under the segments of the ordinate.

Fig. 7.

From any point H, in the curve, let HE be drawn parallel to the diameter VN, to meet its ordinate GQ in E; then $P \times HE = GE \times EQ$.

^a 7.^b 8. 2.

For, let the semi-ordinate HK be drawn; then $GN^2 = P \times VN^a$, and $HK^2 = P \times VK^a$, therefore $GN^2 - HK^2 = P \times KN$, or $\overline{GN + HK} \times \overline{GN - HK}^b = P \times HE$, that is $P \times HE = GE \times EQ$.

COR. 1. The parameter of any diameter is to the sum of two semi-ordinates as their difference to the difference of their abscisses.

COR. 2. Straight lines, drawn parallel to a diameter, from points in the curve, to meet any chord, are to

one

COYIC SECTIONS

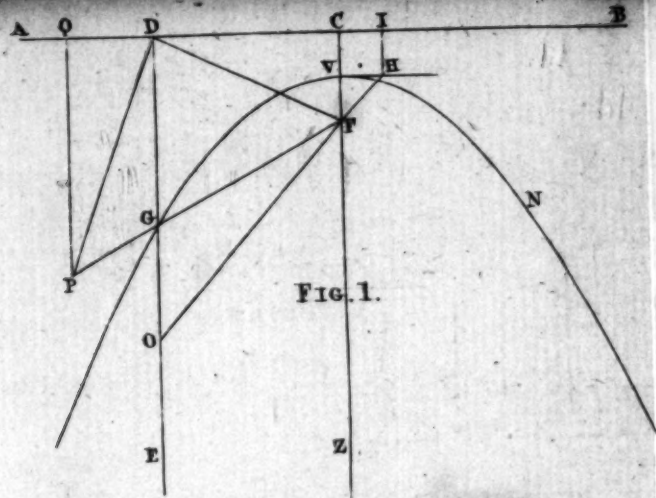


Fig. 1.

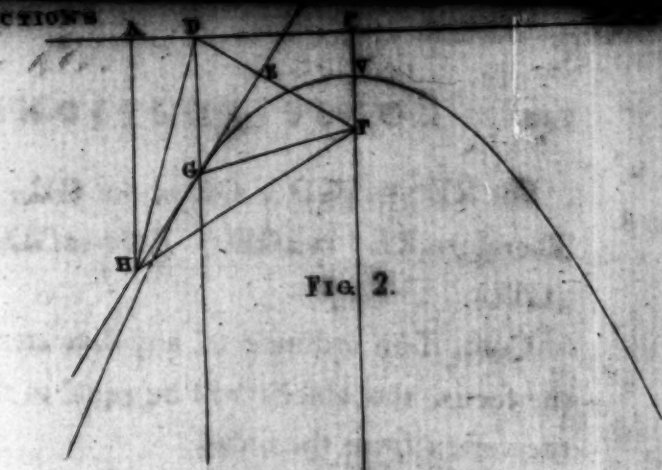


FIG. 2

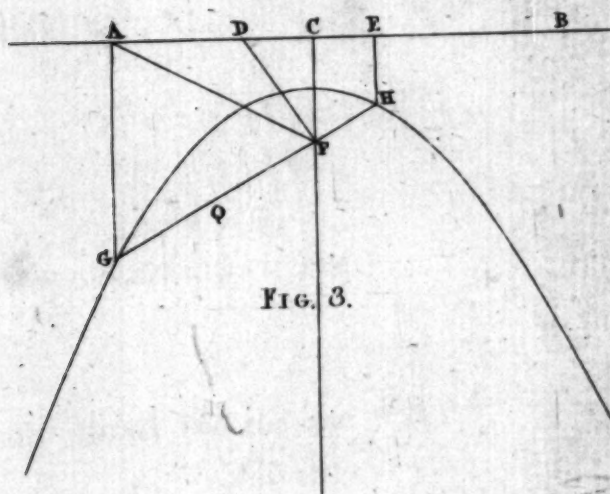


FIG. 3.

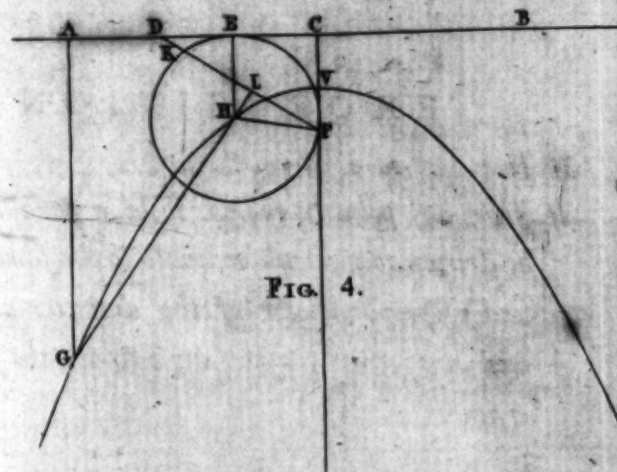


FIG. 4.

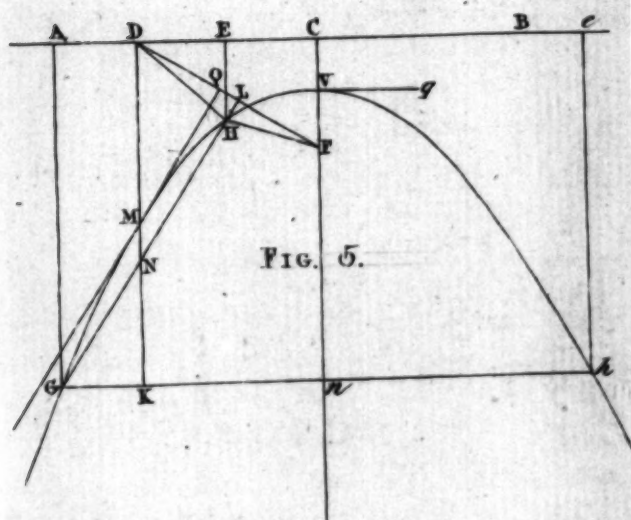


FIG. 5.

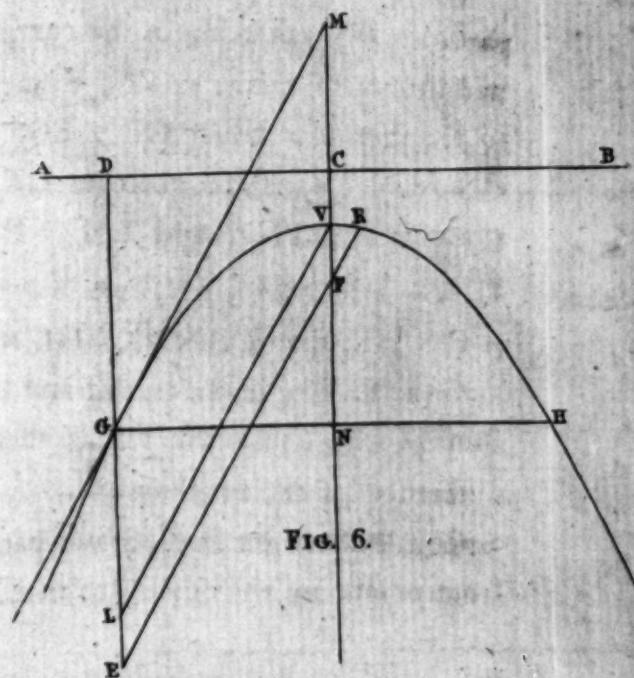


FIG. 6.

one another as the rectangles under the segments of the chord.

COR. 3. If two parallel chords meet any diameter, the rectangles under their segments will be to each other directly as the parts of the diameter which they intercept from the vertex.

PROPOSITION XI.

If a diameter be cut by any straight line passing through two points in the parabola, the part intercepted from the vertex will be a mean proportional between the abscisses corresponding to the two ordinates drawn from the same points in the curve.

Let G, H be two points in the curve, and the line GH cut the diameter VN in O; also, let the semi-ordinates GN, HK be drawn; then $VN : VO :: VO : VK$. Fig. 7. 8.

For $VN : VK (:: GN^2 : HK^2 :: GO^2 : HO^2) :: ON^2 : OK^2$; therefore $VN : VO :: VO : VK$. ^a 4. c. 7.
^d 16. 5.
cor. ib.

COR. If two tangents, and the line joining their points of contact, meet the same diameter, the segment of the diameter, intercepted by this line from the vertex, will be a mean proportional between those intercepted by the tangents.

PROPOSITION XII.

Any two chords which intersect each other in the focus of a parabola, are to one another directly as the rectangles under their segments.

Fig. 9:

Let the chords GH and PQ intersect one another in the focus F, then $GH : PQ :: GFH : PFQ$.

9.

10.

For, since GH and PQ are equal to the parameters of the diameters to which they are ordinates^a, $GH \times VF = GFH$ ^b, and $PQ \times VF = PFQ$; consequently, $GH : PQ :: GFH : PFQ$.

COR. If two chords intersect in any point, the rectangles under their segments are to one another as the parameters of those diameters to which they are ordinates.

Let GH cut the diameter VN in O; also let PQ cut the diameter VN in O; then VN : VO :: VO : VO.

APPENDIX.

PROP. 1. Any straight line, drawn from the focus of a parabola to a point in the directrix, is a mean proportional between half the parameters of the diameters which pass through its extremities.

PROP. 2. If, from any point in the parabola, a tangent, semi-ordinate, and perpendicular, be drawn

to

to meet the same diameter, their squares will be to one another as the parameters of three diameters, that which passes through the point of contact, that which they meet, and the axis.

PROP. 3. If, through the focus of a parabola, a semi-ordinate be applied to the axis, and from its extremity a tangent be drawn to meet another semi-ordinate produced; then shall the produced semi-ordinate be equal to the line joining its extremity in the curve and the focus.

PROP. 4. If a tangent be drawn from any point in the curve, to meet the axis produced, and from the point of contact a perpendicular to the tangent, and a semi-ordinate to the axis, be drawn; then the segment of the axis between the perpendicular and semi-ordinate will be equal to half the parameter of the axis, and the segment between the perpendicular and tangent equal to half the parameter of the diameter which passes through the point of contact.

PROP. 5. To find the directrix and focus of a parabola given in position.

PROP. 6. If, from the vertex of any diameter, a straight line be drawn to the extremity of a semi-ordinate meeting another semi-ordinate, the latter will be a mean proportional between its segments next the diameter and the former.

PROP. 7. A diameter of a parabola, the tangent at the vertex of that diameter, and a point in the curve;

curve, being given, to find the directrix and focus.

PROP. 8. If, from any point in a tangent, a parallel to the diameter passing through the point of contact be drawn to meet an ordinate of the same, the rectangle under the parallel, and its external segment, will be equal to the square of that part of it that is between the curve, and the line joining the vertex of the diameter, and either extremity of its ordinate.

PROP. 9. The focus and directrix of a parabola being given, to draw a tangent to the curve parallel to a line given in position that is not perpendicular to the directrix.

PROP. 10. If there be two tangents to a parabola, such, that the straight line joining their points of contact pass through the focus, they will cut each other at right angles, their intersection will be in the directrix, and the line joining it with the focus will be perpendicular to the line joining the points of contact.

PROP. 11. From a given point, in a given parabola, to draw a tangent without finding the focus.

PROP. 12. Two parabolas of equal parameters, having their axes in the same line, and in the same direction, but their vertices at different points, being produced indefinitely, continually approach, but never meet.

PROP.

PROP. 13. In a given parabola to find a diameter that makes a given angle with its ordinates.

PROP. 14. If, from any point in a tangent, a straight line be drawn, to meet the curve and the diameter passing through the point of contact, the square of its segment, between the tangent and the diameter, will be equal to the rectangle under its segments between the tangent and the points in the curve.

PROP. 15. The focus and directrix of a parabola being given, to draw a tangent to the curve from a given point without it.

PROP. 16. If, from a point in a parabola, a semi-ordinate be applied to a diameter, and from the same point any other straight line be drawn to meet it, the square of this line will be equal to the rectangle under the absciss of that diameter, and the parameter of the diameter to which the straight line, when produced, is an ordinate.

CONIC

CONIC SECTIONS.

B O O K II.

E L L I P S E.

D E F I N I T I O N S.

1. **L**ET there be two given points, and, in the same plane, let a point move around them in such a manner, that the sum of its distances from them may be always the same, it will describe a curve called an ellipse.

2. The given points are named the foci, and the middle of the line that joins them, the center of the ellipse.

3. The distance of the center from one of the foci is called the excentricity.

4. A diameter is a straight line drawn through the center, and terminated on both sides by the curve.

5. The

5. The diameter which passes through the foci is named the transverse axis, and that which is perpendicular to it, the conjugate axis.

6. An ordinate to a diameter is a straight line not passing through the center, but terminated by the curve, and bisected by the diameter.

7. Two diameters are said to be conjugate to one another when each is parallel to the ordinates of the other.

8. The parameter of a diameter is a third proportional to that diameter and its conjugate.

PROPOSITION I.

If, from any point in an ellipse, straight lines be drawn to the foci, their sum is equal to the transverse axis.

Let EFGH be an ellipse, of which EF is the transverse, and GH the conjugate axis, A, B the foci, and D any point in the curve; then $AD + DB = EF$. Fig. 10.

For $EA + EB = FA + FB$; hence $AE = BF$, and def. 1.
 $AD + DB = EA + EB = EF$.

COR. 1. If two straight lines be drawn to the foci, from a point without the ellipse, their sum is greater than the transverse; but, if from a point within it, less.

CON.

COR. 2. A point is without, in, or within the curve, according as the sum of the lines drawn to the foci is greater, equal, or less, than the transverse axis.

COR. 3. The distance of either extremity of the conjugate axis, from one of the foci, is equal to half the transverse.

COR. 4. The transverse and conjugate axis are bisected in the center.

COR. 5. A perpendicular to the transverse at one of its extremities is a tangent to the ellipse.

COR. 6. The square of half the conjugate axis is equal to the rectangle under the segments into which the transverse is divided in one of the foci.

COR. 7. The distance of the foci is a mean proportional between the sum and difference of the transverse and conjugate axis.

PROPOSITION II.

The straight line which bisects the angle adjacent to that which is contained by two straight lines, drawn from any point in the ellipse to the foci, is a tangent to the curve in that point.

Fig. 11.

Let D be any point in the curve, from which AD, DB are drawn to the foci, and let the angle BDI adjacent to ADB, be bisected by the line DT; then is DT a tangent to the ellipse in the point D.

For,

For, let any other point R be assumed in DT , and DI being made equal to DB , let BI , RA , RB , and RI be drawn. The line DNT , which bisects the vertical angle of the isosceles triangle BDI , also bisects the base BI at right angles. Hence $RB = RI^a$, and $AR + RB = AR + RI$, and therefore greater than AI^b or EF^c . Consequently the point R is without the ellipse d .

^a 4. 1.

^b 20. 1.

^c 1.

^d 2. c. 1.

COR. 1. A perpendicular to the conjugate axis, at one of its extremities, is a tangent to the ellipse.

COR. 2. The method of drawing a tangent from a given point in the curve, also, of drawing a tangent parallel to a line given in position, is evident.

COR. 3. There cannot be more than one tangent to the ellipse at the same point. For D is the point in the line DT , the sum of whose distances from A , B is the least possible: If any other line were drawn through D , a point (different from D) may be found in it, such that the sum of its distances from A , B should be the least possible, and consequently less than $AD + DB$; whence that point would be within the ellipse, and the line passing through it would cut, and not touch the curve.

COR. 4. Every tangent bisects the angle adjacent to that contained by straight lines drawn to the foci from the point of contact, or, which is the same, these lines make equal angles with the tangent.

$D d$

COR.

For,

COR. 5. A straight line drawn from the center to meet a tangent, and parallel to the line joining the point of contact, and one of the foci, is equal to half the transverse axis.

For, since $BC = CA$, and $BN = NI$, the line that joins CN will be parallel to AD , and equal to the half of AI .

COR. 6. A perpendicular to a tangent, from one of the foci, and a parallel to the line joining the other focus, and the point of contact from the center, meet the tangent in the same point.

COR. 7. If a chord pass through one of the foci, and the tangents at its extremities be produced to meet, the straight line that joins the point of concurrence and the focus, will be perpendicular to the chord.

Let the tangents at the extremes of the chord DBO , passing through the focus, meet in the point T , and let TI be perpendicular to AD ; then, because DT, TO bisect the exterior angles of the triangle ADO , AI is equal to half its perimeter $^* = AD + DO$. Hence $DB = DI$, and TDI, TDO equal in every respect.

* 2. c. 19. 3.

& 2. c. 13.

3.

PROPO.

PROPOSITION III.

From any point in an ellipse, a perpendicular being let fall upon the transverse, and straight lines drawn to the foci, as half the transverse is to the excentricity, so is the distance of the center from the perpendicular to half the difference of the straight lines drawn to the foci.

Let D be the point in the curve, from which DA, Fig. 12. DB are drawn to the foci, and DK perpendicular to EF, then $CF : CB :: CK : \frac{AD - DB}{2}$.

For, let $EL = AD$, then $LF = DB$, and $2CL = AD - DB$. But $AD + DB \times AD - DB = AB \times 2CK$, ^{a 5. 2.} or $EF \times 2CL = AB \times 2CK$, therefore $EF : AB :: 2CK$ ^{b 11. 2.} $: 2CL$, and $CF : CB :: CK : CL$.

PROPOSITION IV.

Any two chords which pass through one of the foci of an ellipse, are to one another directly as the rectangles under their segments.

Let DO, MI be the two chords passing through the Fig. 13. focus B, $DO : MI :: DB \times BO : MB \times BI$.

For,

For, let the tangents at D, O meet each other in R, through C let NQ be drawn parallel to DO, meeting the tangents in N, Q, and let RB meet NQ in P. It has been proved that CN or $CQ = CF^a$; therefore $NQ = EF$; that $CF : CB :: CK : CL^b$, therefore $CF \times CL = CB \times CK$, and that DBR , or DBP is a right angle^c, therefore the triangles DBK , BCP , are similar, and $DB \times CP = BC \times BK$. Hence $CF^2 = CF \times CL + CF \times LF^d = BC \times CK + DB \times CQ$
 $= BC \times CK + BC \times BK + BD \times PQ = BC^2 + DB \times PQ$, and $DB \times PQ = CF^2 - BC^2 = EB \times BF^e$.
 But $DO : NQ :: BO : PQ :: DB \times BO : DB \times PQ$:
 Therefore $DO : EF :: DB \times BO : EB \times BF$

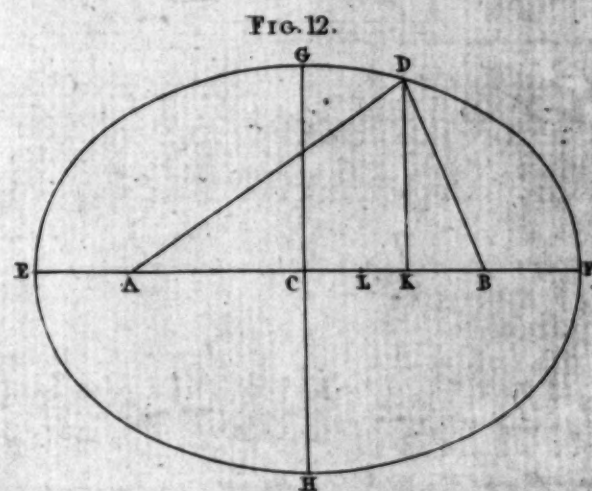
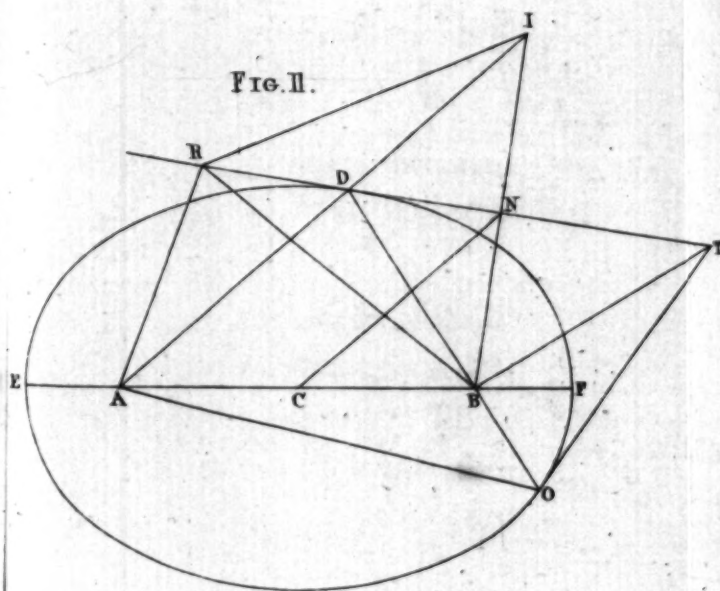
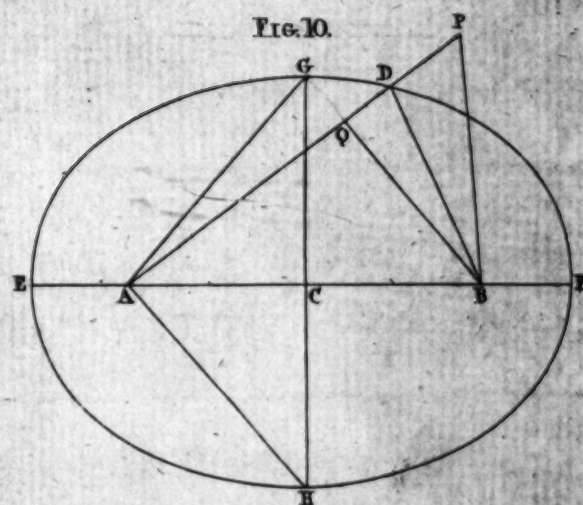
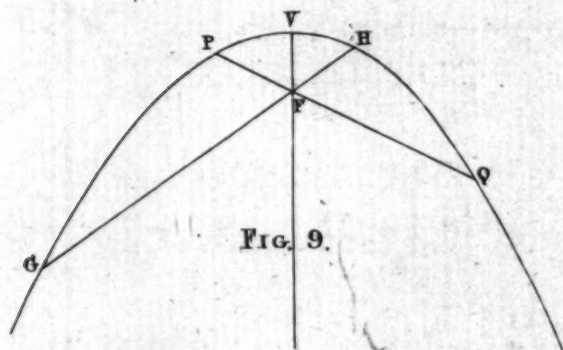
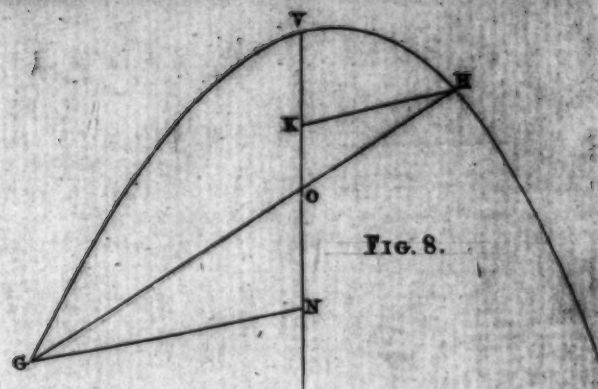
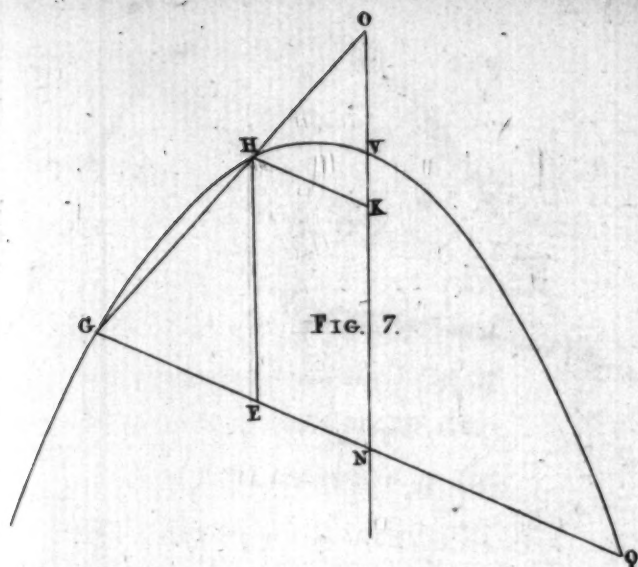
In like manner $EF : MI :: EB \times BF : MI \times BI$
 Consequently $DO : MI :: DB \times BO : MB \times BI$:

COR. The rectangle under any chord passing thro' the focus and the parameter of the transverse, is equal to four times the rectangle under its segments.

For, since $EF : DO :: EB \times BF : DB \times BO$, if P be the parameter of the transverse, $EF \times P : DO \times P :: 4EB \times BF : 4DB \times BL$. But $EF \times P = *GH^2$
 $= 4EB \times BF^b$; therefore $DL \times P = 4DB \times BO^c$.

PROPO.

• That EF, GH are conjugate diameters, will be proved independent of this Cor.



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PROPOSITION V.

If a tangent to the ellipse meet either axis produced, and a perpendicular to the axis be drawn from the point of contact, then shall the semi-axis be a mean proportional between its segments intercepted from the center by the tangent and perpendicular.

CASE I. Let the tangent DT, and the perpendicular DK, meet the transverse in T, K; then $CT : CF :: CF : CK$. Fig. 14.

For, since DT bisects the angle BDI, $AT : TB :: AD : DB^a$, and $\frac{AT+TB}{2} : \frac{AT-TB}{2} :: \frac{AD+DB}{2} : \frac{AD-DB}{2}$. 3. 5.

That is, $CT : CB :: CF : CL$

But $CB : CF :: CL : CK^c$

Therefore $CT : CF :: CF : CK^d$

^b 12. 4.
& 1. 4.

^c 3.

^d 9. 4.

CASE II. Let the tangent DT, and the perpendicular DN, meet the conjugate in M, N; then $MC : CG :: CG : CN$.

For, let DO, perpendicular to DT, meet the transverse in O, then DO evidently bisects the angle ADB, therefore $AO : OB :: AD : DB^a$, hence $AT : TB :: AO : OB$, and $CT : CB :: CB : CO^b$, therefore

CT

$CT \times CO = CB^2$, but $CT \times CK = CF^2$, consequently
 $CT \times KO = CF^2 - CB^2 = CG^2$.

* 1. c. 12.
 3. & 4. 5.

Again, from the similar triangles CTM, KDO*,
 $CT \times KO = CM \times DK = CM \times CN$, therefore CM
 $\times CN = CG^2$, and $CM : CG :: CG : CN$.

COR. 1. The rectangles under the segments of the
 axis, and the segments of the semiaxis produced, both
 being divided by the perpendicular, are equal.

For $EKF = CF^2 - CK^2 = TCK - CK^2 = CKT$.

COR. 2. The segments into which the axis is di-
 vided by the tangent, are directly proportional to the
 segments into which it is divided by the perpendicu-
 lar.

For, since $CT : CF :: CF : CK$, $CT + CF : CT -$
 $CF :: CF + CK : CF - CK$, that is, $ET : TF : EK$
 $: KF$.

COR. 3. The segments CT, CB, CO, are propor-
 tional; and, in like manner, it may be proved that
 CK, CL, CO, are proportional.

PROPOSITION VI.

*If, from any point in an ellipse, a perpendicular fall upon
 either axis, as the square of that axis is to the square
 of the other, so is the rectangle under the segments of
 the former, to the square of the perpendicular.*

Let

Let the perpendicular DK fall from any point in Fig. 16. the curve upon EF, then $EF^2 : GH^2 :: EKF : DK^2$.

For, since $CT : CF :: CF : CK^a$, $CF^2 : CK^2 :: a^2 : 5$. $CT : CK^b$, and, by conversion, $CF^2 : EKF :: CT : TK$. ^{b 3. c. 10. 5.} But $CT : TK :: CG^2 : CN^2^b$, or DK^2 , therefore $CF^2 : EKF :: CG^2 : DK^2$, and alternately $CF^2 : CG^2 :: EKF : DK^2$. Consequently $EF^2 : GH^2 :: EKF : DK^2^c$. If the perpendicular DN fall upon GH, it ^{c 1. 4.} may be proved in the same manner that $GH^2 : EF^2 :: GNH : DN^2$.

COR. 1. The squares of perpendiculars, falling from points in the curve, upon either axis, are to one another as the rectangles under the segments into which they divide the axis.

COR. 2. If a circle be described upon the transverse, Fig. 15. and another upon the conjugate axis, the former will contain, and the latter will be contained in the ellipse, so that the curve lines shall touch one another only in the extremities of their common diameter.

COR. 3. The two axes are the greatest and least diameters of the ellipse.

COR. 4. If a circle be described upon either axis of an ellipse, perpendiculars to the common diameter are cut proportionally by the curves.

COR. 5. Every straight line terminating in the curve, and parallel to one axis, is an ordinate to the other; and conversely.

COR.

COR. 6. The two axes are conjugate diameters.

COR. 7. The ordinate through the focus is the parameter of the transverse.

COR. 8. Equal ordinates to either axis are equally distant from the center; the greater ordinate is nearer the center; and conversely.

Fig. 23.

COR. 9. If, from any point in the curve, a straight line be drawn to the conjugate axis, equal to half the transverse, its segment intercepted from the same point by the transverse, will be equal to half the conjugate; and conversely.

PROPOSITION VII.

A straight line, not passing through the center, but terminated by the ellipse, and parallel to a tangent, is an ordinate to the diameter that passes through the point of contact.

Fig. 17.

The chord MN, parallel to the tangent at D, is an ordinate to the diameter CD.

For, let MN meet CT in L, and from the points M, N, let MI, NO be drawn perpendicular to EF, MR, NQ parallel to DC. Then, by similar triangles, $MI : DK :: RI : CK$.
And $MI : DK :: IL : KT$.

Hence

Hence^a $EIF : EKF^b :: RIL : CKT$ ^a 4. c. 9. 5.
^b 1. c. 6.
^c 1. c. 5.
 But $EKF = CKT^c$
 Therefore $EIF = RIL$
 And $EI : IL :: RI : IF$
 By composition $EL : IL : RF : IF$
 By alternation and
 conversion $EL : ER + FL :: IL : FL$
 In like manner $EL : EQ + FL :: OL : FL$
 Hence^d $IL : OL :: EQ + FL : ER + FL$ ^d 2. c. 9. 4.
 And $RL : QL :: EQ + FL : ER + FL$
 By division $RQ : QL :: RQ : ER + FL$.

Consequently $QL = ER + FL$, $ER = FQ$, $CR = CQ$,
 and $MP = PN$.

COR. 1. Every ordinate to a diameter is parallel to
 the tangent at its vertex.

For, if not, let a tangent be drawn parallel to it,
 then the diameter through the point of contact would
 bisect it; and thus the same line would be bisected in
 two different points, which is absurd.

COR. 2. All the ordinates to the same diameter are
 parallel to one another.

COR. 3. A straight line, drawn through the vertex
 of a diameter, parallel to its ordinate, is a tangent to
 the curve in that point.

COR. 4. The diameter which bisects one of two
 parallel chords, will also bisect the other.

COR. 5. The straight line which bisects two parallel
 chords, and terminates in the curve, is a diameter.

COR. 6. If two tangents be at the vertices of the same diameter, they are parallel; and conversely.

COR. 7. Every diameter divides the ellipse into two equal parts.

PROPOSITION VIII.

Two diameters, one of which is parallel to the tangent, at the vertex of the other, are conjugate to one another.

Fig. 18.

Let the diameter PQ be parallel to the tangent DT, at the vertex of the diameter DI; then shall PQ, DI be conjugate diameters.

For, let DK, QN be perpendicular to EF, and let the tangents at D, Q meet EF in TR. The triangles TDK, CQN, by reason of parallel lines, are equiangular, therefore $TK : CN :: DK : QN$; hence $TK^2 : CN^2 :: DK^2 : QN^2 :: EKF : ENF^a :: CKT : RNC^b$, and $TK : CN :: CK : RN$, or $DK : QN :: CK : RN$.

^a I. C. 6.

^b I. C. 5.

Consequently the triangles DKC, RNQ, are equiangular, and DC parallel to RQ. Whence PQ, DI are each parallel to the ordinates of the other^c, that is, they are conjugate diameters.

^c I. C. 7.

COR. 1. If, from the extremities of two conjugate semi-diameters, perpendiculars be let fall upon either axis, the rectangle under the segments, into which the axis is divided by one of the perpendiculars, is equal

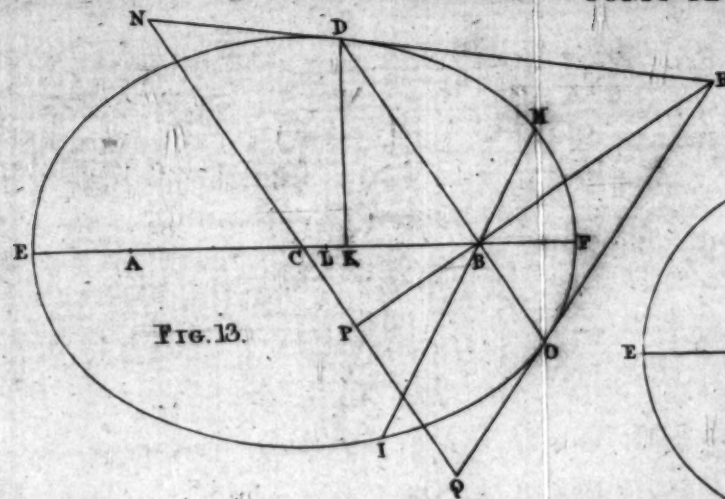


FIG. 13.

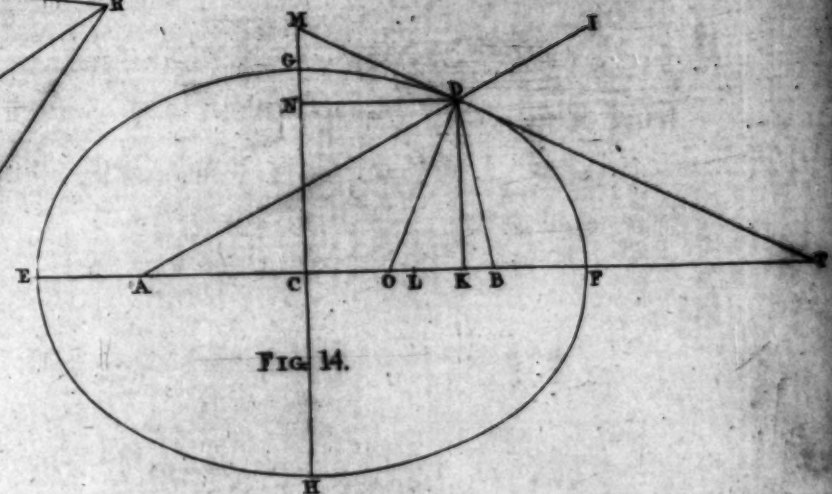


FIG. 14.

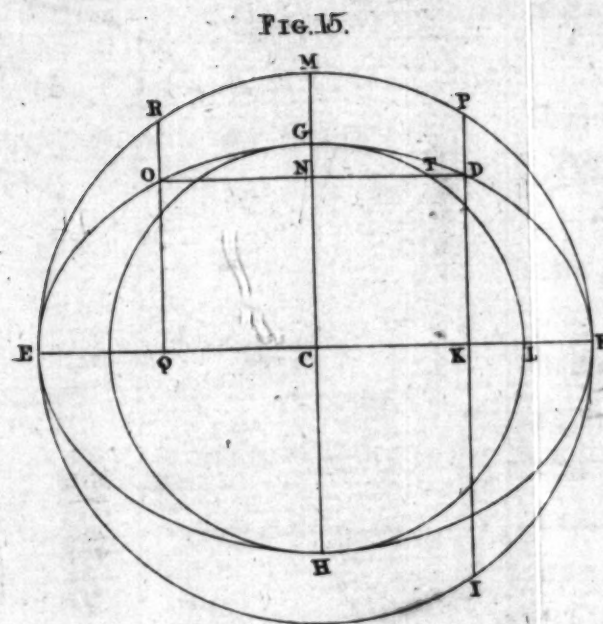


FIG. 15.

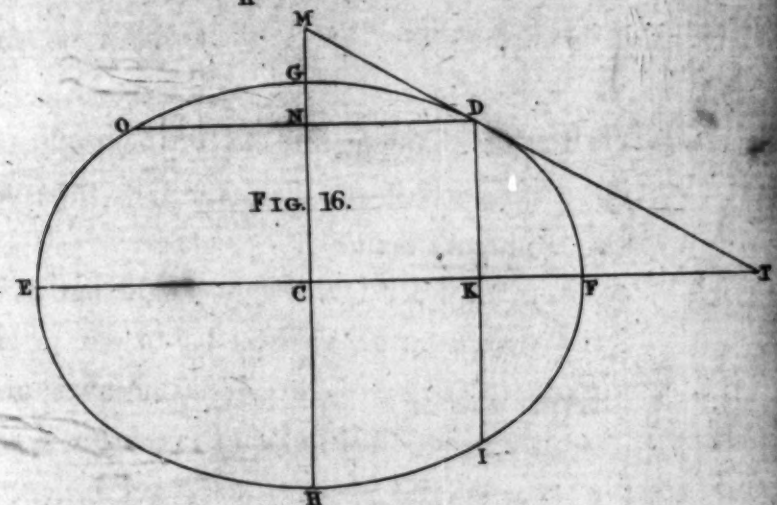


FIG. 16.

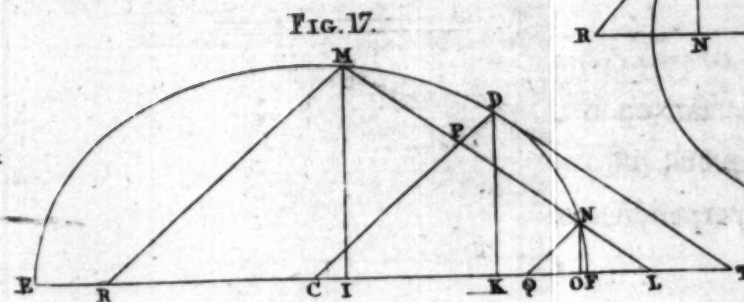


FIG. 17.

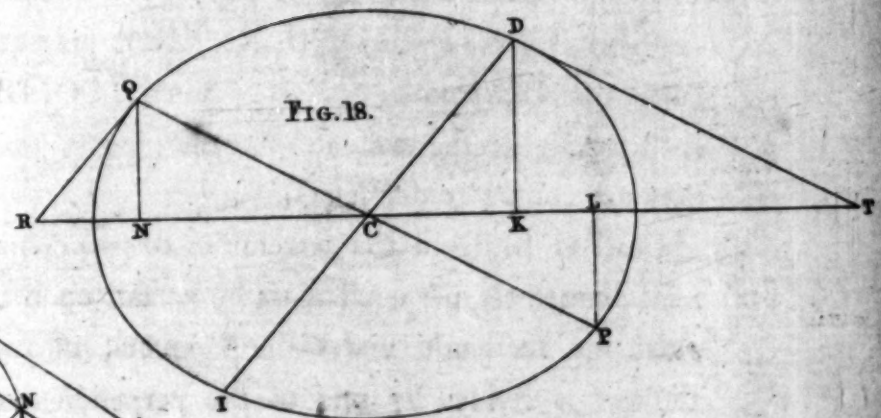


FIG. 18.

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equal to the square of the segment, which the other intercepts from the center.

For, since RQ is parallel to CD, $CT : TK :: CR : CN$, and $CT \times CK : CK \times KT :: CR \times CN : CN^2$. But $CT \times CK = CF^2 = CR \times CN$, therefore $CK \times KT$, or $EK \times KF = CN^2$.

COR. 2. Also, the sum of the square of the segments, intercepted from the center, is equal to the square of half the axis upon which the perpendiculars fall; and the sum of the squares of the perpendiculars is equal to the square of half the other axis.

COR. 3. The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the transverse and conjugate axis.

PROPOSITION IX.

Any semi-diameter which meets an ordinate and tangent drawn from the same point in the ellipse, is a mean proportional between the segments intercepted from the center.

CASE I. Let the semi-diameter CN meet a tangent Fig. 19. and ordinate drawn from F, one extremity of either axis in M, P: Then $CM : CN :: CN : CP$.

For, let the tangent and ordinate from N meet the axes EF in T, K. It has already been demonstrated, with

- ^a 5. with respect to either axis, that $CT : CF :: CF : CK^a$;
hence $CM : CN :: CF : CK :: CT : CF :: CN : CP$.

CASE II. Let the semidiameter CN meet a tangent and ordinate from any other point D in M, P, $CM : CN :: CN : CP$.

For, let DP, the tangent at N, DM, and NR parallel to DM, meet the axis EF in L, Q, R, T. Also, let DK, NO, be the semi-ordinates to EF from D, N. The lines DK, DT, DL, are respectively parallel to NO, NR, NQ.

- ^b 2. c. 6. Hence $CO \times OQ : CK \times KT (::^b NO^2 : DK^2) :: OQ^2 : KL^2$

& 1. c. 5. By alternation $CO : OQ :: CK \times KT : KL^2$

But $OQ : OR :: KL : KT$

Therefore $CO : OR :: CK : KL$

By conversion $CO : CR :: CK : CL$

- ^c 5. Again $CT \times CK = CF^2 = CQ \times CO^c$

Therefore $CT : CO :: CQ : CK$

Hence $CT : CR :: CQ : CL$

And $CM : CN :: CN : CP$

COR. 1. Tangents, at the extremities of an ordinate, meet its diameter produced in the same point.

COR. 2. A straight line, drawn through the center, from the point of intersection of two tangents, bisects the line that joins the points of contact.

Fig. 22. 23. COR. 3. If a circle be described upon any diameter of an ellipse, and through any point of it an ordinate of

of each curve be drawn, tangents, at their extremes, will meet one another in the common diameter produced.

COR. 4. The segments into which any diameter is divided by a tangent, are directly proportional to the segments into which it is divided by an ordinate from the point of contact.

COR. 5. If a semi-diameter be produced to meet a tangent, the rectangle under the segments into which it is divided, by an ordinate from the point of contact, is equal to the rectangle under the segments, into which the whole diameter is divided by the same ordinate.

PROPOSITION X.

If a semi-ordinate be applied to any diameter of an ellipse, the square of that diameter is to the square of its conjugate, as the rectangle under its segments to the square of the semi-ordinate.

Let PQ, DI be two conjugate diameters, and MN Fig. 21.
any semi-ordinate to DI ; $DI^2 : PQ^2 :: DNI : MN^2$.

For, let the tangent at M meet DI, PQ in R, O,
and let ML be a semi-ordinate to PQ. Then, because
CR

9.

$CR : CD :: CD : CN^a$, $CD^2 : CN^2 :: CR : CN$, and
 $CD^2 : DNI :: CR : RN^b :: OC : CL :: CQ^2 : CL^2$.

11. 4.

Hence $CD^2 : CQ^2 :: DNI : MN^2$, and $DI^2 : PQ^2 :: DNI : MN^2$.

COR. 1. The squares of semi-ordinates are to one another as the rectangles under the segments of the diameter.

COR. 2. Any diameter of an ellipse is to its parameter as the rectangle under its segments to the square of the semi-ordinate that divides them.

PROPOSITION XI.

If, through any point in the ellipse, a parallel to the conjugate axis meet the transverse, and the focal tangent, its segment between these will be equal to the distance of that point from the focus.

Fig. 24.

At the focus B, let BD be a semi-ordinate to the transverse; then DT, touching the ellipse in D, is named the focal tangent. Through any point R in the curve, let RQ, parallel to GC, meet DT, EF in P, Q; PQ is equal to RB.

For, let $FL = RB$.

5.
3.

Then, because $CT : CF :: CF : CB^a :: CQ : CL^b$

Alternately $CT : CQ :: CF : CL$

By conversion $CT : TQ :: CF : FL$

Hence

Hence $MC : PQ :: CF : RB$.

But $MC = CF$, consequently $PQ = RB$.

^c 5. c. 2.

COR. 1. A perpendicular to the transverse, at either extremity, limited by the focal tangent, is equal to the distance of that extremity from the focus.

COR. 2. The distance of any point in the curve, from the perpendicular to the transverse, at the point where it intersects the focal tangent, is to its distance from the focus as half the transverse is to the eccentricity.

PROPOSITION XII.

If four straight lines touch an ellipse at the vertices of two conjugate diameters, the parallelogram which they contain will be equal to the rectangle under the transverse and conjugate axis.

Let DI, PQ be two conjugate diameters, and LM, MN, NO, OL tangents at their vertices. Then $LMNO$ is a parallelogram^a, divided into four equal parallelograms by PQ, DI ; and the whole MO is equal to the rectangle under EF and GH , or the part $CN = CF \times CG$.

^a 6. c. 7.

For, let DK, PR be perpendicular to EF , and let EF meet the tangent DN in T , and TP be joined.

Then,

^b 6.^c 1. c. 8. 5.^d 5. c. 9.^e 5.^f 9. 4.

Then, since^b $CF^2 : CG^2 :: (ERF^c =) CK^2 : PR^2$

$$CF : CG :: CK : PR^d$$

But $CT : CF :: CF : CK^e$

Therefore $CT : CG :: CF : PR^f$

Hence $CF \times CG = CT \times PR =$ double the triangle CTP = the parallelogram CN.

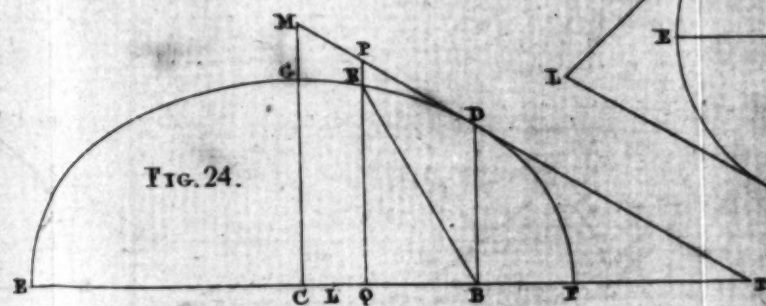
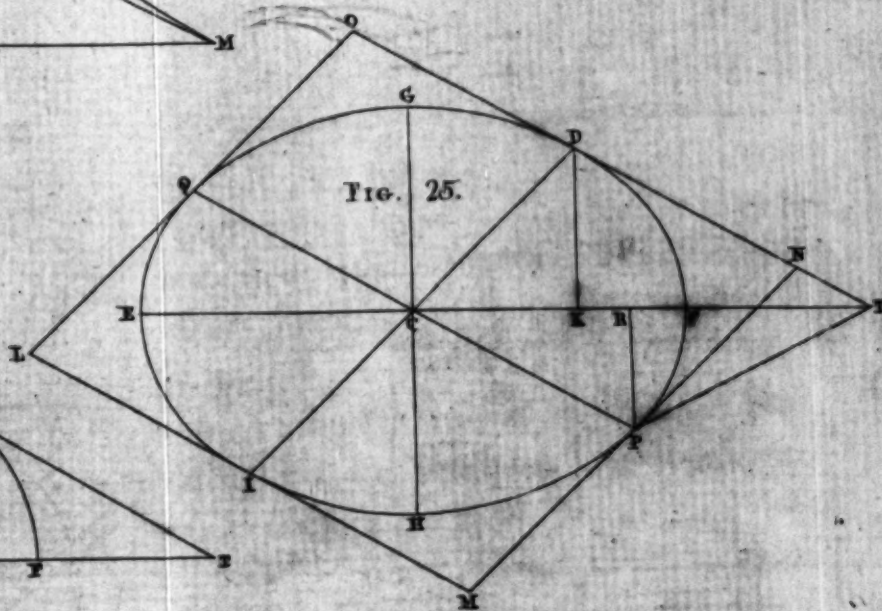
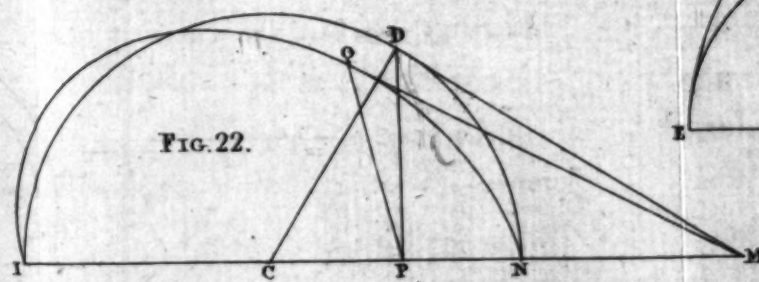
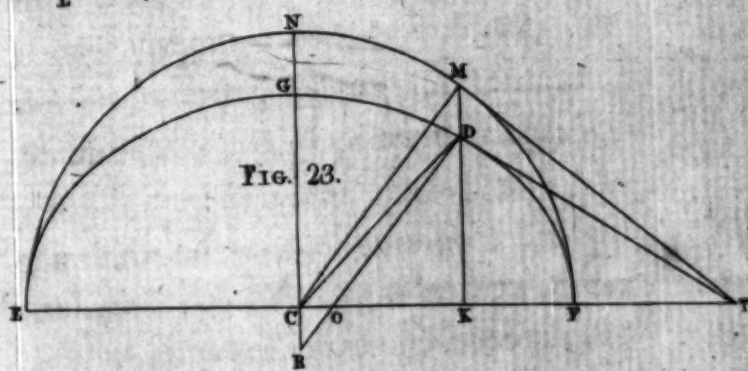
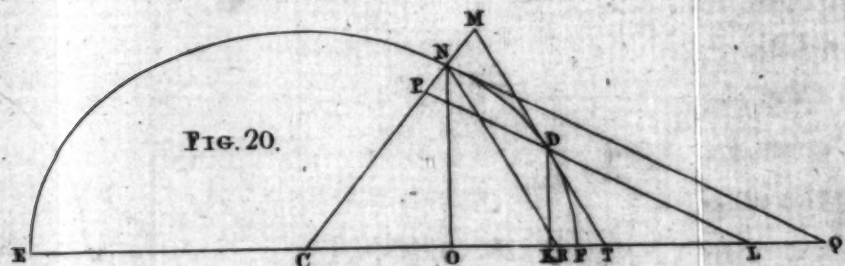
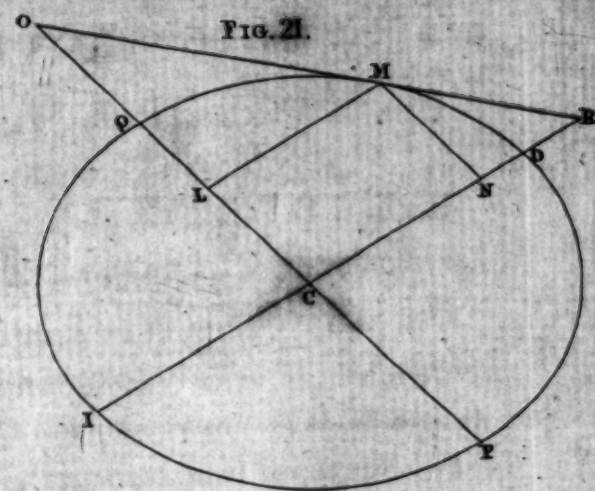
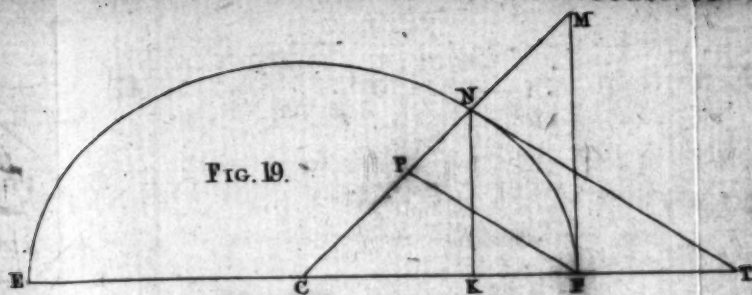
COR. 1. If perpendiculars fall from the extremities of two conjugate semi-diameters upon either axis, the axis upon which they fall is to the other as the segment intercepted by one of the perpendiculars from the center is to the other perpendicular.

COR. 2. All parallelograms whose sides touch an ellipse in the vertices of two conjugate diameters, are of the same magnitude.

COR. 3. Every parallelogram whose angular points are the vertices of two conjugate diameters, is equal to half the rectangle under the axis.

A P P E N D I X.

PROP. 1. The rectangle under perpendiculars to the transverse passing through its extremities, the rectangle under perpendiculars from the foci to any tangent, the rectangle under perpendiculars to the transverse, from the center and the point of contact, and the rectangle under perpendiculars to the tangent, from the point



point of contact, and the center, (all the perpendiculars being limited by the transverse and tangent), are each equal to the square of the semiconjugate axis.

PROP. 2. If, from any point in an ellipse, two straight lines be drawn through the extremities of a diameter, any other straight line parallel to the tangent at that point, and terminating in these lines, will be bisected by an ordinate to that diameter, from the point of contact.

PROP. 3. If the diagonals of a quadrilateral figure, inscribed in an ellipse, intersect one another in one of the foci, and two opposite sides be produced to meet, the straight line that joins the point of concurrence and the foci will bisect the angle formed by these diagonals.

PROP. 4. If two tangents, at the vertices of a diameter, meet any third tangent, the rectangle under the two former, and the rectangle under the segments of the latter, from the point of contact, are respectively equal to the squares of the semi-diameters to which the tangents are parallel.

PROP. 5. If, through the extremities of the transverse axis, two tangents be drawn to meet a third, the circle described upon the intercepted tangent will pass through the foci.

PROP. 6. If, from the vertices of two conjugate semi-diameters, two semi-ordinates be applied to any diameter, the square of its segment, between the center and either semi-ordinate, will be equal to the rectangle under the segments into which it is divided by the other semi-ordinate.

PROP. 7. If, from the vertex of any diameter, straight lines be drawn to the foci, their rectangle is equal to the square of its semi-conjugate.

PROP. 8. If a tangent meet two conjugate diameters, the rectangle under its segments, from the point of contact, will be equal to the square of the semi-diameter to which it is parallel.

PROP. 9. Every diameter is a mean proportional between the ordinate to its conjugate, which passes through the focus and the transverse axis.

PROP. 10. If, from any point, in one tangent, a straight line be drawn parallel to another, to meet the curve in two points, the rectangle under its segments, between the tangent and the curve, is equal to the square of its segment between the tangent and the line joining the points of contact.

PROP. 11. The rectangles under the segments of two chords which intersect, are to one another directly as the squares of the diameters to which they are parallel.

PROP.

PROP. 12. If, from any point, in a given ellipse, straight lines, parallel to lines given in position, be drawn to meet the sides of a given quadrilateral inscribed in it, the rectangles under those drawn to opposite sides, will have to each other a given ratio.

CONIC

CONIC SECTIONS.

B O O K III.

H Y P E R B O L A.

D E F I N I T I O N S.

Fig. 26. 1. **L**ET there be two given points, and let any point in the line which joins them, except the middle, move in the same plane with these, in such a manner, that the difference of its distances from the given points may be always the same, it will describe a curve, called an hyperbola. And every two points that are equidistant from the middle describe two equal and similar curves, called opposite hyperbolas.

2. The given points are named the foci; that part of the line that joins them, intercepted by opposite hyperbolas,

hyperbolas, the transverse axis; and the middle of the same line, the center.

3. The conjugate axis is a straight line, passing thro' the center, perpendicular to the transverse, and limited by a circle described from one extremity of the transverse with the distance of either focus from the center as a radius.

4. If two other hyperbolas be described, having the conjugate for their transverse axis, and their foci at the same distance from the center as the foci of the former, these are also called opposite hyperbolas, and have the transverse of the former for their conjugate axis.

5. A diameter is a straight line drawn through the center, and terminated by opposite hyperbolas.

6. An ordinate to a diameter is a straight line, terminated by the hyperbola, and bisected by that diameter produced.

7. An external ordinate to a diameter is a straight line, terminated by opposite hyperbolas, and bisected by that diameter, or the same produced.

8. Two diameters are said to be conjugate to one another when they are mutually parallel to each other's ordinates.

9. A third proportional, to any diameter, and its conjugate, is called its parameter.

10. An asymptote of the hyperbola is a straight line, which, being produced indefinitely, does not meet,

meet, but continually approaches the curve, so as to come within less than any given distance from it.

PROPOSITION I.

If, from a point in any hyperbola, straight lines be drawn to the foci, their difference is equal to the transverse axis.

Fig. 27.

Let DF, EP be opposite hyperbolas, of which A, B are the foci, C the center, EF the transverse axis, and D any point in one of the curves; then $AD - DB = EF$.

^a def 1.
^b 5. 2.

For $AD - DB = AF - FB^a = 2CF^b = EF$.

COR. 1. If two straight lines be drawn to the foci, from a point without the opposite hyperbolas, their difference is less than the transverse axis, but, if from any point within either of them, it is greater.

Let M be a point within the hyperbola DF, and N a point without, between the curve and its conjugate axis. The lines AM, NB necessarily meet the curve; let them meet it in the points D, D; then $AM - MB$ is greater than $AM - \overline{MD} + \overline{DB}$, or EF, but $AN - NB$ less than $\overline{AD} + \overline{DN} - NB$, or EF.

COR. 2. A point is within, in, or without the curve, according as the difference of its distances from the foci, is greater, equal, or less, than the transverse axis.

COR.

COR. 3. The conjugate axis is bisected in the center.

COR. 4. The rectangle under the segments, into which the transverse is divided in one of the foci, is equal to the square of the semi-conjugate axis.

COR. 5. The square of the distance of the foci is equal to the sum of the squares of the transverse and conjugate axis. This, and the two preceding Corollaries follow from the third definition.

COR. 6. A perpendicular to the transverse, at its extremity, is a tangent to the hyperbola.

PROPOSITION II.

The straight line which bisects the angle, contained by two straight lines, drawn from any point in the hyperbola to the foci, is a tangent to the curve in that point.

Let D be any point in the hyperbola DFO, from Fig. 28. which AD, DB are drawn to the foci, and let the angle BDI be bisected by the line DT, then is DT a tangent to the curve in the point D.

For, let any other point R be assumed in DT, and DI being made equal to DB, let BI, RA, RB, RI be drawn. The line DNT, which bisects the vertical angle of the isosceles triangle BDI, also bisects the base BI at right angles. Hence $RB = RI$, and
AR

AR — RB = AR — RI, and therefore less than AI or EF. Consequently, the point R is without the hyperbola.

* 2. c. 1.

COR. 1. The method of drawing a tangent from a given point in the curve, also, of drawing a tangent parallel to a line given in position not parallel to the transverse, is evident.

COR. 2. There cannot be more than one tangent to the hyperbola at the same point.

COR. 3. Every tangent bisects the angle contained by straight lines drawn to the foci from the point of contact; or, which is the same, these lines make equal angles with the tangent.

COR. 4. Every tangent to the same hyperbola meets the transverse between its vertex and the center.

COR. 5. A straight line drawn from the center to meet a tangent, and parallel to the line joining the point of contact, and one of the foci, is equal to half the transverse axis.

COR. 6. A perpendicular to a tangent, from one of the foci, and a parallel to the line joining the other focus, and the point of contact from the center, meet the tangent in the same point.

COR. 7. If a chord pass through one of the foci, and the tangents at its extremities be produced to meet, the straight line that joins the point of concurrence and the focus, will be perpendicular to the chord.

Let

Let the tangents at the extremes of the chord DBL, passing through the focus, meet in the point O, and let OI be perpendicular to AD : Then, because $AD - DB = AL - LB$, $AD + LB = AL + DB =$ half the perimeter of the triangle ADL $= AL + DI$, because O is the center of the inscribed circle. Hence $DB = DI$, and the triangles DBO, DIO equal in every respect.

PROPOSITION III.

From any point in an hyperbola, a perpendicular being let fall upon the transverse, and straight lines drawn to the foci ; as half the transverse is to the distance of the center from the focus, so is the distance of the center from the perpendicular to half the sum of the straight lines drawn to the foci.

Let D be the point in the curve, from which DA, DB are drawn to the foci, and DK perpendicular to EF; then $CF : CB :: CK : \frac{AD + DB}{2}$. Fig. 27.

For, let $EFL = AD$, then $LF = DB$, and $2CL = AD + DB^a$. But $\overline{AD + DB} \times \overline{AD - DB} = AB \times ^a 5. 2.$
 $2CK^b$, or $EF \times 2CL = AB \times 2CK$; therefore $EF : ^b 11. 2.$
 $AB :: 2CK : 2CL$, and $CF : CB :: CK : CL$.

PROPOSITION IV.

Any two chords which intersect each other in the focus of an hyperbola, are to one another directly as the rectangles under their segments.

Fig. 29.

Let the chords DO, MI intersect one another in the focus B of the hyperbola DFO, $DO : MI :: DB \times BO : MB \times BI$.

For, let the tangents at D, O meet each other in R, through C let NQ be drawn parallel to DO, meeting the tangents in N, Q, and let BR meet NQ in P. It has been proved that CN or $CQ = CF^a$,
^{a 5. c. 2.} therefore $NQ = EF$; that $CF : CB :: CK : CL^b$,
^{b 3.} therefore $CF \times CL = CB \times CK$, and that DBR is
^{c 7. c. 2.} a right angle^c, therefore the triangles DBK, BCP are similar, and $DB \times CP = BC \times BK$. Hence
 $CF^2 = CF \times CL - CF \times FL = BC \times CK - DB \times CQ = BC \times CK - BC \times BK - DB \times PQ = BC^2 - DB \times PQ$, and $DB \times PQ = BC^2 - CF^2 = EB \times BF$.
 But $DO : NQ :: BO : PQ :: DB \times BO : DB \times PQ$.
 Therefore $DO : EF :: DB \times BO : EB \times BF$,
 In like manner, $EF : MI :: EB \times BF : MB \times BI$.
 Consequently $DO : MI :: DB \times BO : MB \times BI$.

COR. The rectangle under any chord passing thro' the focus and the parameter of the transverse, is equal to four times the rectangle under its segments.

PROPO.

CONIC SECTIONS Book II

FIG. 26.

PROPOSITION IV

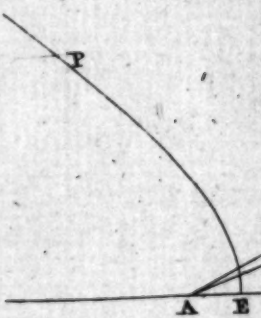
Let two chords intersect each other in the focus of a parabola, are to one another directly as the squares of the segments.

Let the chords DO , MI intersect one another in the focus B of the parabola DO , $DO \cdot MI : DB \cdot BO : MB \times BI$.

For, let the tangents at D , O meet each other at R , through C let NO be drawn parallel to DO , meeting the tangents in N , O , and let BR meet NO in P . It has been proved that $CN = CO = CP$.

FIG. 27.

Therefore $NO = EP$; that $CF : CB :: CK : CL$ therefore $CF \times CL = CB \times CK$, and that DBK a right angle, therefore the triangles DBK , BCP are similar, and $DB \times CP = BC \times BK$. Hence $CF \times CL - CF \times CL = BC \times CK - DB \times BK$.

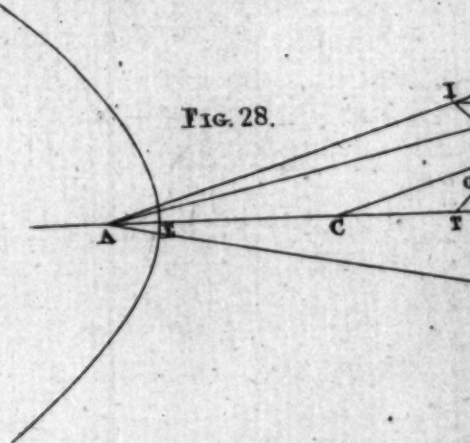


$CO = BC \times CK - BC \times BK - DB \times BK = BC \times BK - DB \times BK$, and $DB \times PO = BC \times BK - CF \times CL = DB \times BK$. Therefore $DO : NO :: BO : PO :: DB \times BO : DB \times PO$.

FIG. 29.

Consequently $DO : MI :: DB \times BO : MB \times BI$. Cor. The rectangle under any chord passing through the focus and the parameter of the transverse is equal to four times the rectangle under its segments.

FIG. 28.



PROPOSITION

PROPOSITION V

If a tangent meet either axis of the hyperbola, and a perpendicular to that axis be drawn from the point of contact; then shall half the axis be a mean proportional between its segments intercepted from the center by the tangent and perpendicular.

CASE I. Let the tangent DT, and perpendicular DK, meet the transverse in T, K, CT : CF :: CF : CK

For, since DT bisects the angle ADB, AT : TB :: AD : DB, and $\frac{AT + TB}{2} : \frac{AT - TB}{2} :: \frac{AD + DB}{2} : \frac{AD - DB}{2}$

That is, CB : CT :: CL : CF.
But CF : CB :: CK : CL.
Therefore CF : CT :: CK : CF.
And CT : CF :: CF : CK.

CASE II. Let the tangent DT, and the perpendicular DN meet the conjugate in M, N, MC : CG :: CG : CN.

For, let DO, perpendicular to DT, meet the transverse in O, DO evidently bisects the exterior vertical angle of the triangle ADB, therefore AO : OB :: AD : DB :: AT : TB. Hence CB : CO :: CT : CB, and

PROPOSITION V.

If a tangent meet either axis of the hyperbola, and a perpendicular to that axis be drawn from the point of contact, then shall half the axis be a mean proportional between its segments intercepted from the center by the tangent and perpendicular.

CASE I. Let the tangent DT, and perpendicular Fig. 30.
DK, meet the transverse in T, K, $CT : CF :: CF :$
CK.

For, since DT bisects the angle ADB, $AT : TB ::$
 $AD : DB$, and $\frac{AT + TB}{2} : \frac{AT - TB}{2} :: \frac{AD + DB}{2}$
 $: \frac{AD - DB}{2}$.

That is, $CB : CT :: CL : CF$.

But $CF : CB :: CK : CL$.

Therefore $CF : CT :: CK : CF$

And $CT : CF :: CF : CK$.

CASE II. Let the tangent DT, and the perpendi-
cular DN meet the conjugate in M, N, $MC : CG ::$
 $CG : CN$.

For, let DO, perpendicular to DT, meet the trans-
verse in O, DO evidently bisects the exterior vertical
angle of the triangle ADB, therefore $AO : OB ::$
 $AD : DB :: AT : TB$. Hence $CB : CO :: CT : CB$,
and

and $CT \times CO = CB^2$. But $CT \times CK = CF^2$, consequently $CT \times KO = CB^2 - CF^2 = CG^2$.

Again, from the similar triangles CTM, DKO, $CT \times KO = MC \times DK = MC \times CN$; therefore $MC \times CN = CG^2$, and $MC : CG :: CG : CN$.

COR. 1. The rectangle under the segments of the transverse, into which it is divided by the perpendicular, is equal to the rectangle under the distances of the perpendicular from the center, and the point where the tangent meets the transverse.

COR. 2. The segments into which either axis is divided by the tangent are directly proportional to the segments into which it is divided by the perpendicular.

PROPOSITION VI.

If, from any point in an hyperbola, a perpendicular fall upon the transverse axis, as the square of that axis is to the square of the other, so is the rectangle under the segments of the former, to the square of the perpendicular.

Fig. 30.

Let the perpendicular DK fall from any point D in the curve, upon EF, $EF^2 : GH^2 :: EKF : DK^2$.

For, since $CT : CF :: CF : CK^2$, $CF^2 : CK^2 :: CT : CK$, and, by conversion, $CF^2 : EKF :: CT : TK$. But
CT

CT : TK :: CG² : CN², or DK², therefore CF² :
EKF :: CG² : DK², and alternately CF² : CG² ::
EKF : DK². Consequently EF² : GH² : EKF : DK².

COR. 1. The square of the conjugate axis is to the square of the transverse, as the sum of the squares of the semi-conjugate, and the distance of the center from a perpendicular falling upon the conjugate, from any point in the curve, to the square of that perpendicular.

COR. 2. The squares of perpendiculars, falling from points in the curve upon its transverse axis, are to one another as the rectangles under the segments of the axis ; and the squares of perpendiculars upon the conjugate, are to one another as the sums of the squares of the semi-conjugate, and the distance of each from the center.

COR. 3. Every straight line, terminating in the hyperbola, or in opposite hyperbolas, and parallel to one axis, is an ordinate to the other ; and conversely.

COR. 4. The two axes are conjugate diameters.

COR. 5. The ordinate through the focus is the parameter of the transverse.

COR. 6. Equal ordinates to either axis are equally distant from the center, the less ordinate is nearer the center ; and conversely.

PROPOSITION NO VIII

If, through any point in the hyperbola, a parallel to the conjugate axis meet the transverse and the focal tangent, its segment between these will be equal to the distance of that point from the focus.

Fig. 31.

Through any point R in the curve, let RQ, parallel to GC, meet the focal tangent DE, and transverse axis EF in P, Q; then $PQ = RB$.

For, let $FL = RB$. Then, because $CT : CF :: CF : CB^a :: CQ : GL^b$.

Alternately $CT : CQ :: CF : CL$

By conversion $CT : TQ :: CF : FL$

Hence $MC : PQ :: CF : RB$

But $MC = CF$, consequently $PQ = RB$.

COR. 1. A perpendicular to the transverse, at one extremity, limited by the focal tangent, is equal to the distance of that extremity from the focus.

COR. 2. The distance of any point in the curve, from the perpendicular to the transverse, at the point where that axis meets the focal tangent, is to its distance from the focus, as half the transverse is to the distance of the center from the focus.

COR. 3. Every perpendicular to the transverse beyond its vertices meets the curve on each side of the axis.

PROPO.

PROPOSITION VIII.

Straight lines drawn through the center, parallel to the lines which join the extremities of the axes, are asymptotes to the hyperbolas.

The lines PQ, RS, drawn through C, parallel to Fig. 32.
FH, HE, do not meet, but continually approach the curve, so as to come within less than any given distance, from it.

For, let FQ be perpendicular to EF, and through the points K, q, assumed in the transverse produced, let IS, *mo* be drawn parallel to GH, meeting PQ in I, *m*, the curve in D, *n*, and RS in S, *o*. Then, since $CF^2 : CG^2 :: EKF : DK^2$, and $FQ = CH$, or $\cdot 6$.
 $CG, CF^2 : FQ^2 :: EKF : DK^2$. Hence $CK^2 : KI^2 :: EKF : DK^2$; but CK^2 is always greater than EKF , therefore KI is greater than DK . Thus, every point of PQ is without the curve.

Again, by alternation and conversion,

$$CK^2 : CF^2 :: KI^2 : IDS$$

$$\text{But } CK^2 : CF^2 :: KI^2 : FQ^2$$

Therefore $IDS = FQ^2$. In like manner, $mo = PQ^2$.

Hence $ID : mn :: mo : DS$, and, since *mo* is greater than DS , ID is greater than mn , whence the point *n* is nearer to PQ than the point D.

Lastly,

Lastly, to show that the distance of the curve from PQ, when both are produced, becomes less than any given line, however small, let Dy be drawn parallel to PQ, at the distance of the given line from it, on the same side with the hyperbola, and meeting FQ in y; then, to Qy and FQ, let DS, parallel to FQ, and terminating in Dy, CS, be a third proportional, and let it be produced to meet PQ in I; ID is equal to Qy, therefore $ID : FQ :: FQ : DS$, and $IDS = FQ^2$. Whence, it is plain, from what has been demonstrated, that D is a point in the curve. Now, D is at the distance of the given line from PQ; consequently the curve, at any point beyond it, is at a less distance from PQ.

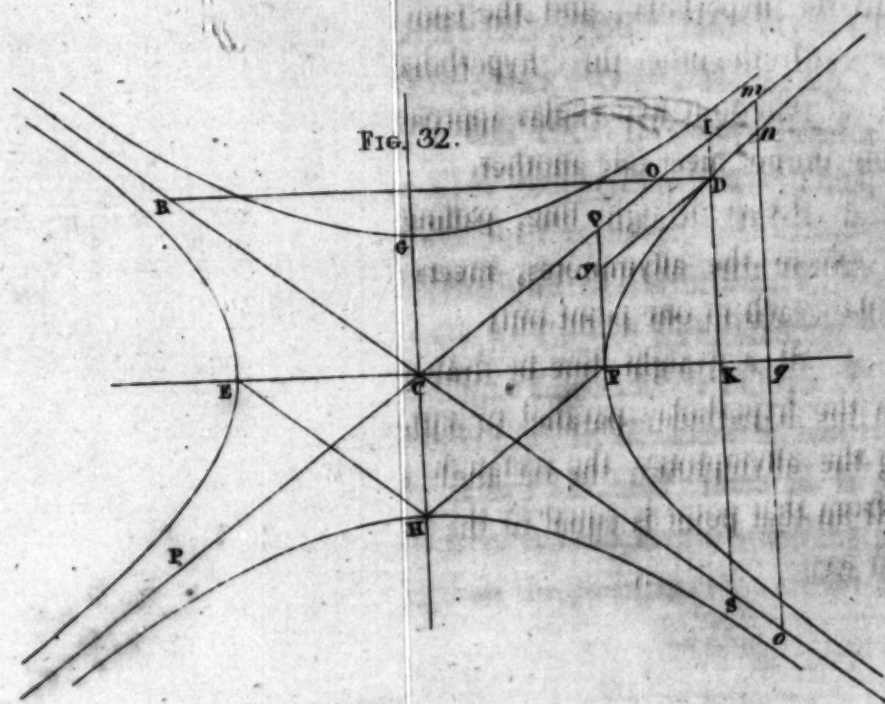
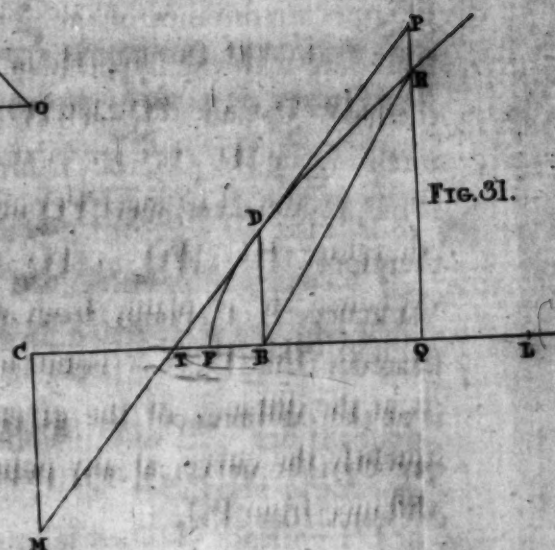
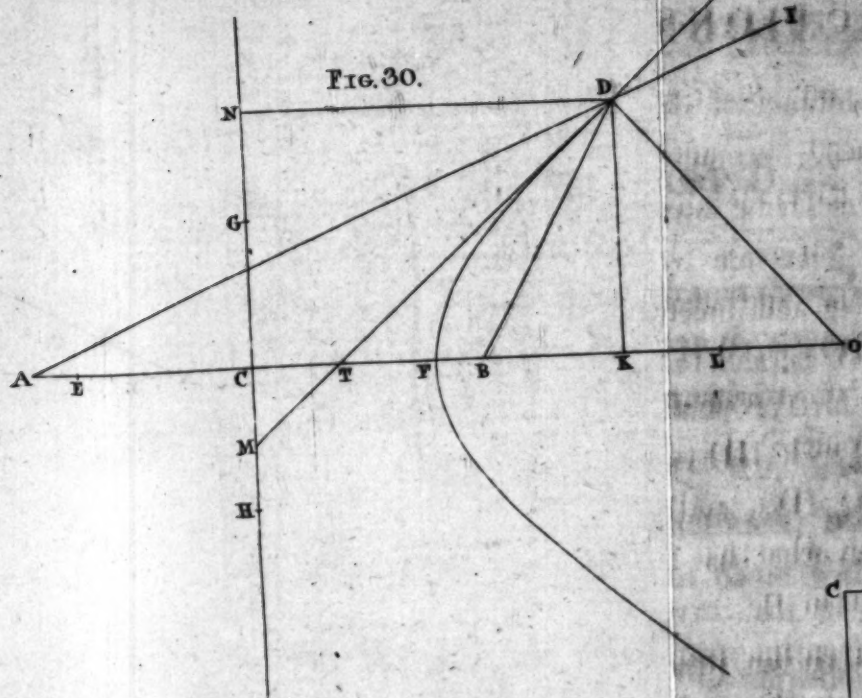
COR. 1. There cannot be more than two assymptotes to the hyperbola; and the same lines are also assymptotes to the other three hyperbolas.

COR. 2. The four hyperbolas approach indefinitely near, but do not meet one another.

COR. 3. Every straight line, passing through the center, except the assymptotes, meets two opposite hyperbolas, each in one point only.

COR. 4. If a straight line be drawn through any point in the hyperbola, parallel to either axis, and meeting the assymptotes, the rectangle under its segments from that point is equal to the square of half the other axis.

For,



For $IDS = FQ^2 = CG^2$; and it may be proved in the same manner that $RDO = CF^2$.

COR. 5. According as the conjugate axis is greater, equal, or less, than the transverse, the angle of the asymptotes is greater, equal, or less, than a right angle. When it is a right angle, the hyperbola is named an equilateral or rectangular hyperbola.

PROPOSITION IX.

If, from any point in an hyperbola, a straight line be drawn parallel to any diameter, to meet the asymptotes, the rectangle under its segments, from that point, is equal to the square of half the diameter.

Let D be any point in the hyperbola, and DO drawn parallel to the diameter LCM, meet the asymptotes in Q, V. Then $VDQ = \frac{1}{4} LM^2$. Fig. 33.

For, through D, and either extremity of LM, let IDS, PLR, be drawn parallel to GH, to meet the asymptotes. Then, because of similar triangles,

$$ID : DQ :: PL : LC$$

And $DS : DV :: LR : LC$

Therefore $IDS : DQ \times DV :: PLR : LC^2$. But $IDS = PLR^2$; therefore $DQ \times DV = LC^2$. In the same manner, $DV \times DQ = CM^2$. Hence $LC = CM$, and $DQ \times DV =$ the square of half the diameter LM. 4 c. 8.

H h

COR.

COR. 1. Every diameter is bisected in the center.

COR. 2. If a straight line be drawn through two points, in the same, or in opposite hyperbolas, its segments from these points, intercepted by the asymptotes, will be equal.

For, if DV meet the curve in O, $OQ \times OV = LC^2 = DV \times DQ$, therefore $OQ : DQ :: DV : VO$, and, $OD : DQ :: OD : VO$. Whence $DQ = VO$, and $DV = QQ$.

COR. 3. Every tangent limited by the asymptotes is bisected in the point of contact, and is equal to that diameter to which it is parallel. Also, conversely, a straight line, terminated by the asymptotes, and having its middle point in the curve, is a tangent to the hyperbola.

COR. 4. If, from two points in the same, or in opposite hyperbolas, parallel straight lines be drawn, to meet the asymptotes, the rectangles under their segments will be equal.

PROPOSITION X.

A straight line, terminated by the hyperbola, and parallel to a tangent, is an ordinate to the diameter that passes through the point of contact.

Fig. 34. O The chord DO, parallel to the tangent XLY, is an ordinate to the diameter LM.

For,

For, let the asymptotes meet the tangent in X, Y , and the chord in Q, V ; also, let LM meet DO in R . Then, because $XL = LY$, $RQ = RV$. But, $DQ = OV$. Consequently $RD = RO$.

COR. 1. A straight line, terminated by opposite hyperbolas, and parallel to a tangent, is an external ordinate to the diameter that passes through the point of contact.

COR. 2. Every ordinate, and every external ordinate, to a diameter, is parallel to the tangent at its vertex.

COR. 3. All ordinates to the same diameters are parallel to one another.

COR. 4. A straight line, drawn through the vertex of a diameter, parallel to its ordinates, is a tangent to the hyperbola.

COR. 5. The diameter which bisects one of two parallel chords, will also bisect the other.

COR. 6. The straight line, which bisects two parallel chords, passes through the center of the hyperbola.

COR. 7. If two tangents be at the vertices of the same diameters, they are parallel; and conversely.

PROPO. I. If a straight line, terminated by the hyperbolas, be parallel to the tangent at the point of contact, it shall be an external ordinate to the diameter that passes through the point of contact.

PROPOSITION XI.

Two diameters, one of which is parallel to the tangent at the vertex of the other, are conjugate to one another.

Fig. 34-

Let LCM and ICN be two diameters, of which IN is parallel to the tangent XY, at the vertex of LM; IN, LM are conjugate diameters.

For, let XI be drawn and produced to meet the other asymptote in Z. Then, because XL is equal and parallel to IC^a, XZ is parallel to LM. Now, the triangles ICZ, LCY, by reason of parallel lines, 3. c. 7. are equiangular, and they have the side IC = LY^a; therefore IZ = LC = IX. Whence XZ is a tangent at I^a; and LM has been proved parallel to it. Consequently, IN, LM are mutually parallel to each other's ordinates.

COR. 1. Tangents to the four hyperbolas, at the vertices of two conjugate diameters, form a parallelogram, whose diagonals are coincident with the asymptotes.

COR. 2. Any two conjugate diameters of an equilateral hyperbola are equal to one another.

For, the angle of its asymptotes being a right

2. c. 9. 3. angle, CL = LX^b = CI.

PROPO-

PROPOSITION XII.

If an ordinate be applied to any diameter, the square of that diameter is to the square of its conjugate as the rectangle under its segments to the square of the semi-ordinate.

Let LCM be any diameter, IN its conjugate, and Fig. 34.
DRO an ordinate to LM, in the hyperbola DLO;
 $LM^2 : IN^2 :: MRL : DR^2$.

For, let DO meet the asymptotes in Q, V, and the tangent at L meet them in X, Y. Then, from similar triangles, $CL^2 : CR^2 :: LX^2 : RQ^2$, by conversion and alternation, $CL^2 : LX^2 :: MRL : RQ^2 - LX^2$. But $LX^2 = CI^2 = QDV^2$, and $RQ^2 - LX^2 = DR^2$; therefore $CL^2 : CI^2 :: MRL : DR^2$, and $LM^2 : IN^2 :: MRL : DR^2$.

COR. 1. If an external ordinate be applied to any diameter, the square of that diameter is to the square of its conjugate, as the sum of the squares of the semi-diameter and segment intercepted by the external ordinate, from the center, to the square of the external semi-ordinate.

COR. 2. The squares of semi-ordinates are to one another as the rectangles under the segments of the diameter.

COR.

COR. 3. Any diameter of an hyperbola is to its parameter, as the rectangle under its segments to the square of the semi-ordinate that divides them.

COR. 4. In an equilateral hyperbola, the rectangle under the segments of any diameter is equal to the square of the semi-ordinate that divides them; and the square of an external semi-ordinate is equal to the sum of the squares of the semi-diameter and segment intercepted from the center.

PROPOSITION XIII.

If from any point in an hyperbola, two straight lines be drawn, to meet the asymptotes, and from any other point in the same, or in the opposite hyperbola, straight lines, parallel to these, be also drawn to the asymptotes; the rectangle under the former shall be equal to the rectangle under the latter.

Fig. 35.

From the point D, in the hyperbola, let there be drawn any two straight lines DQ, DR to terminate in the asymptotes CQ, CS, and from any other point N let NP be drawn parallel to DQ, and NS parallel to DR, to terminate in the same lines. Then $RDQ = PNS$.

For, let LDK, and MNO be drawn parallel to the conjugate axis, to meet the asymptotes. Then, from the

the similar triangles RDK , SNO , $RD : DK :: NS : NO$, and from the similar triangles, LDQ , MNP , $DQ : LD :: PN : MN$. Therefore $RDQ : LDK :: PNS : MNO$. But $LDK = MNO$. Consequently $RDQ = PNS$.

COR. 1. If, from two points, in the same, or in opposite hyperbolas, straight lines be drawn, each parallel to one asymptote, and meeting the other, they are to one another inversely as the parts they intercept from the center. Fig. 36.

COR. 2. If, from any point in a given hyperbola, two straight lines be drawn parallel to the asymptotes, the parallelogram formed thereby is of a given magnitude.

COR. 3. Every sector of an hyperbola is equal to the quadrilateral figure contained by the curve, by one asymptote, and by parallels to the other, through the extremities of the base of the sector.

For, since the parallelograms RQ , PS are equal, the two triangles CQD , CSN , are together equal to PS , and these equals being taken from the figure $CQDNS$, there remains the sector CDN equal to the quadrilateral figure $PQDN$.

PROPO.

PROPOSITION XIV.

If, in one of the asymptotes of an hyperbola, any number of points be assumed, such, that their distances from the center be in continued proportion, and straight lines be drawn from these points to the curve, parallel to the other asymptote, the quadrilateral figures formed thereby will be equal.

Fig. 37.

Let the points A, B, D, E, be assumed in the asymptote CS, so that $CA : CB :: CB : CD :: CD : CE$, and let AF, BG, DH, EK, be drawn to meet the curve parallel to the other asymptote CQ; the quadrilateral figures AFGB, BGHD, DHKE shall be equal.

* 3. c. 9.
* 2. c. 9.

For, let the tangent at G, and the line that joins H, F meet CQ in N, L, and CS in O, M. Then, because $OG = GN^a$, $OB = BC$, and because $MH = FL^b$, $MD = AC$; therefore $MD : OB :: CB : CD :: DH : BG$. Hence, the triangles MDH, OBG, which have the angles at D and B equal, are equiangular, and LM is parallel to NO. The diameter CG, therefore, bisects the chord HF, and every chord parallel to HF^c. Consequently, CG bisects the segment FGH of the hyperbola. But it also bisects the triangle FCH;

* 10.

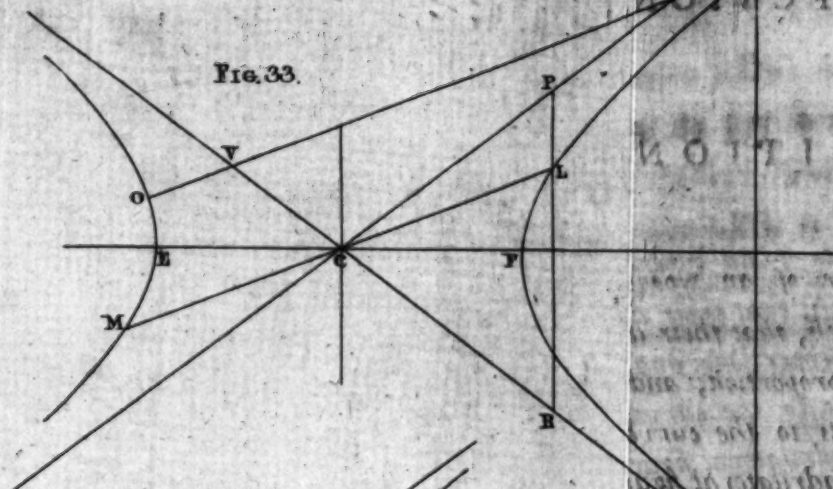


Fig. 33.

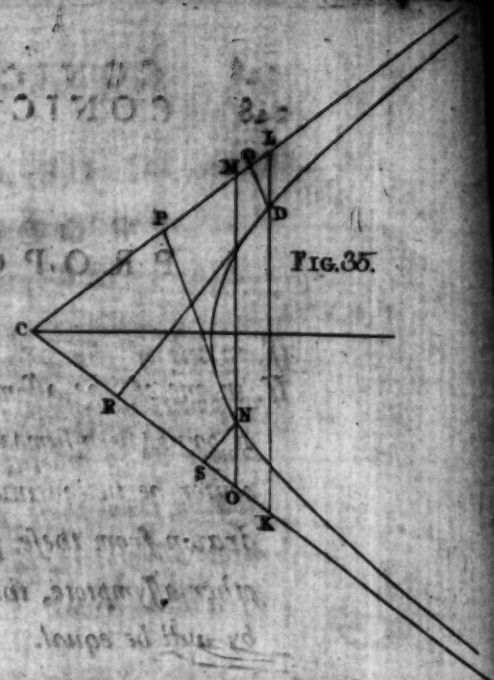


Fig. 35.

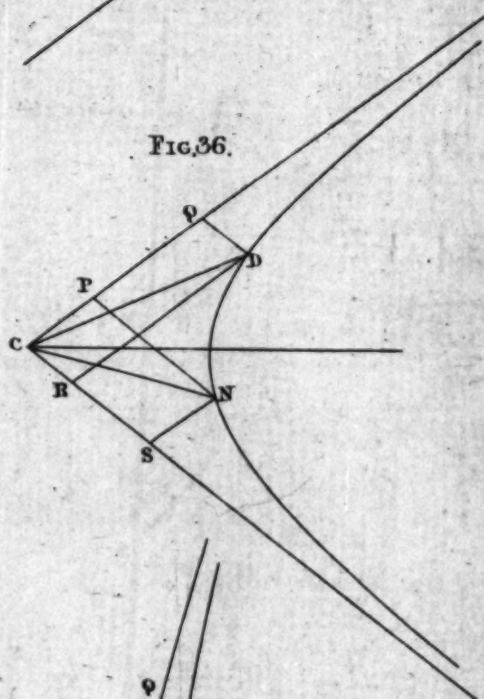


Fig. 36.

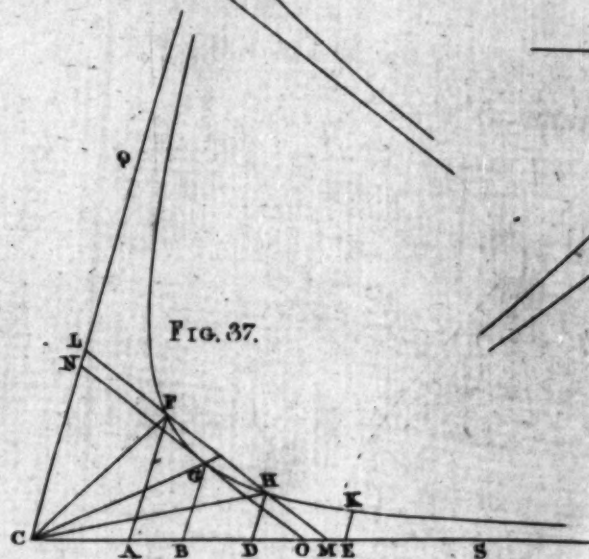


Fig. 37.

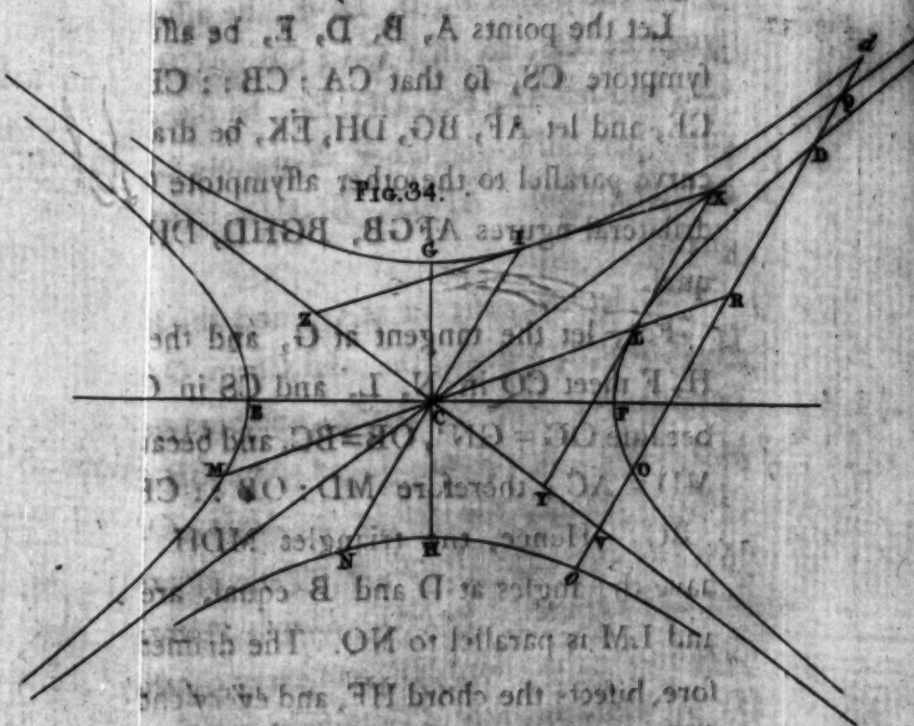


Fig. 34.

Let the points A, B, D, E, be all
 (symptote CS, to that CA, CB, CI
 CI, and let AF, BG, DH, EK, be the
 lines parallel to the other asymptote
 the lines AFGB, BGHD, DH
 let the center of G, and the
 let CO be the line L, and CS in C
 CO = GN, OR = BC, and beca
 AC, therefore MD, OR, CI
 the angles at D and B equal, we
 and LM is parallel to NO. The diamet
 fore bisect the chord HF, and beca
 HE, consequently CG bisects the
 of the hyperbola. But in the

SECTION 10

1. The first part of the section is devoted to a description of the general character of the country, and to a statement of the principal features of the topography.

2. The second part of the section is devoted to a description of the principal rivers and streams, and to a statement of the principal features of the hydrography.

3. The third part of the section is devoted to a description of the principal cities and towns, and to a statement of the principal features of the population.

4. The fourth part of the section is devoted to a description of the principal industries and occupations, and to a statement of the principal features of the commerce.

5. The fifth part of the section is devoted to a description of the principal minerals and natural resources, and to a statement of the principal features of the geology.

6. The sixth part of the section is devoted to a description of the principal climate and weather, and to a statement of the principal features of the meteorology.

7. The seventh part of the section is devoted to a description of the principal diseases and health, and to a statement of the principal features of the medicine.

8. The eighth part of the section is devoted to a description of the principal laws and government, and to a statement of the principal features of the politics.

9. The ninth part of the section is devoted to a description of the principal education and literature, and to a statement of the principal features of the culture.

10. The tenth part of the section is devoted to a description of the principal religion and customs, and to a statement of the principal features of the society.

11. The eleventh part of the section is devoted to a description of the principal art and science, and to a statement of the principal features of the progress.

12. The twelfth part of the section is devoted to a description of the principal history and events, and to a statement of the principal features of the past.

13. The thirteenth part of the section is devoted to a description of the principal future and prospects, and to a statement of the principal features of the hope.

FCH; therefore the sector $CFC = CGH$, and the quadrilateral $AFGB = BGHD$.

d 3. c. 13.

N. B. The 9th and 12th proposition of the ellipse may be applied to the hyperbola, and demonstrated in the same manner.

A P P E N D I X.

PROP. 1. The square of any semi-diameter of an hyperbola is equal to the rectangle under the distances of its vertex from the foci, added to the difference of the squares of the semi-transverse, and semi-conjugate axis.

PROP. 2. Every tangent of an hyperbola is harmonically divided by the transverse axis and perpendiculars falling upon it from the foci.

PROP. 3. The difference of the squares of any two conjugate diameters of an hyperbola is equal to the difference of the squares of the two axes.

PROP. 4. If, from any point in an hyperbola, straight lines be drawn through the vertices of a diameter, to limit the tangents at these points, the rectangle under the tangents will be equal to the square of the semi-conjugate diameter.

PROP. 5. A semi-ordinate to any diameter is a mean proportional between its segments, intercepted from the diameter by two straight lines intersecting each other in any point of the curve, and passing through the vertices of the diameter.

PROP. 6. If a quadrilateral figure be formed by tangents to the four hyperbolas, a straight line thro' the center, parallel to that which joins two opposite points of contact, will divide the two opposite sides of the figure, so that the segments of the one shall be inversely proportional to the segments of the other.

PROP. 7. Also, the straight line which joins the middle points of its diagonals will pass through the center of the hyperbolas.

PROP. 8. If, through a fixed point, any straight line be drawn, to meet the hyperbola, or opposite hyperbolas, in two points, the rectangle under its segments, from the fixed point, will be to the rectangle under its segments, intercepted by the asymptotes, from either point in the curve, in a constant ratio.

PROP. 9. If, from a point in one of the asymptotes of an hyperbola, any straight line be drawn to intersect the curve (or opposite curves,) in two points, and from the points of section, lines parallel to the same asymptote be drawn to meet the other, the sum (or

(or difference) of the parallels will always be of the same magnitude.

PROP. 10. A straight line, drawn from any point in the curve, parallel to either asymptote, to terminate in the directrix, is equal to a straight line drawn from the same point to the focus.

N. B. The propositions in the Appendix to the preceding Book may be applied to the hyperbola.

B. O. K. IV.

DEFINITION 2.

1. If there be a circle, and a fixed point, with
out its plane, and let a straight line, always
passing through that point, and indefinitely extended
both ways, revolve about the circle in its circumference
till it reaches the two surfaces thus described, and each of
them named a conic surface; the fixed point is ver-
tex, the circle its base, and the straight line passing
through the vertex and the center its axis.
2. A cone is a solid bounded by a circle and a co-
nic surface. The fixed point is called the vertex, and
the circle the base of the cone. Also, the straight line
passing through the fixed point, and the center of the
circle is named the axis of the cone.

(or difference) of the parallels will always be of the

CONIC SECTIONS.

PROP. 10. A straight line, drawn from any point in the curve, parallel to either asymptote, to terminate in the diameter, is equal to a straight line drawn from the same point to the focus.

N. B. The propositions in the Appendix to the preceding Book may be applied to the hyperbola.

B. O. O. K. IV.

DEFINITIONS.

Fig. 39.

1. **L**ET there be a circle, and a fixed point, without its plane, and let a straight line, always passing through that point, and indefinitely extended both ways, revolve about the circle in its circumference; the two surfaces thus described are each of them named a conic surface, the fixed point its vertex, the circle its base, and the straight line passing through the vertex and the center its axis.

2. A cone is a solid bounded by a circle and a conic surface. The fixed point is called the vertex, and the circle the base of the cone. Also, the straight line passing through the fixed point, and the center of the circle, is named the axis of the cone.

3. Let

3. Let there be a circle, and any straight line intersecting its plane at the center, and let another straight line, always parallel to the former, revolve about the circle in its circumference; the surface thus described is named a cylindric surface, of which the circle is the base, and the straight line through its center the axis. Fig. 4.

4. A cylinder is a solid bounded by two equal and parallel circles, and a cylindric surface. Either of the circles is named the base, and the straight line joining their centers the axis of the cylinder.

5. A cone or cylinder is termed right or oblique, according as the axis is perpendicular or oblique to the base.

6. The section of a cone or cylinder is termed parallel or oblique, according as its plane is parallel or inclined to the base.

7. When a cone or cylinder is cut by a plane touching the axis, and perpendicular to the base, and by another plane perpendicular to the former, in such a manner, that the common section of the two planes make angles with the common sections of the first plane and the surface, alternately equal to those which the common section of the first plane and the base makes with the same lines on the same side, the section of the second plane is called a subcontrary section.

8. When

Fig. 8. When the circumference of a circle, described through any point in the curve of a parabola, hyperbola, or ellipse, so nearly coincides with the curve, that the circumference of no other circle can be described between them, the conic section is said to be of the same curvature with the circle at that point, and the radius of the circle is termed the radius of curvature.

COROLLARIES FROM THE DEFINITIONS.

COR. 1. A straight line joining the vertex, and any point in a conic surface, lies wholly in that surface; its continuation one way is in the same, and its continuation the other way in the opposite surface; also, every such line meets the circumference of the base.

COR. 2. Any plane touching the axis of a conic surface cuts that surface in two straight lines.

COR. 3. If a cone be cut by a plane touching the axis, or by a plane through the vertex, and any two points in the circumference of the base, the section is a triangle.

COR. 4. If a plane, touching a tangent to the base, pass through the vertex, it touches the conic surface in the straight line joining the vertex and the point of

con-

contact, every point in the plane, except in that straight line, being without the surface.

Cor. 5. A straight line, drawn from any point in a cylindric surface, parallel to the axis, lies wholly in that surface.

Cor. 6. Any plane touching the axis of a cylindric surface cuts that surface in two parallel straight lines.

PROPOSITION I.

If a conic or cylindric surface be cut by a plane parallel to the base, their line of common section is the circumference of a circle, having its center in the axis.

11. Let $ABDCA$ be a conic surface, of which BCD is the base, and AE the axis, and let it be cut by the plane GOL , parallel to BCD , the line of common section $GKHL$ is the circumference of a circle having its center in AE .

For, let any two planes, ABC , AFD , touching the axis AE , cut the surfaces in the straight lines AB , AC ; AF , AD ; the base BCD , in the diameters BEC , DEF , and the parallel section GOL , in the straight lines GOH , KOL . Then BC is parallel to GH , and FD to KL . Hence, by similar triangles, $ED : OK$ ($\because AE : AO$) $:: EC : OH$. But $ED = EC$, therefore $OK = OH$. Consequently all straight lines drawn from

from the point O , where the axis meets the parallel plane GOL , to terminate in the line of common section $GKHL$, are equal to one another, and $GKHL$ is the circumference of a circle, of which O is the center.

Fig. 40.

2. Let $MBDCN$ be a cylindric surface, of which BCD is the base, and AE the axis, and let it be cut by the plane GOL , parallel to BCD , the line of common section $GKHL$ is the circumference of a circle, having its center in AE .

cor. 6.

For, let any two planes, ABC , AFD , touching the axis AE , cut the surface in the straight lines MB , NC ; LF , KD , the base BCD , in the diameters BEC , DEF , and the parallel section GOL , in the straight lines GOH , KOL . Then BC is parallel to GH , and FD to KL . But NC , KD are each parallel to AE . Therefore OD , OC are parallelograms, and $OK = ED = EC = OH$. Whence $GKHL$ is the circumference of a circle, of which O is the center.

COR. 1. If a conic surface be cut by a plane, parallel to the base, the solid betwixt that plane and the vertex is a cone.

COR. 2. If a cylindric surface be cut by two planes parallel to the base, the solid betwixt them, and the solids between each of them and the base, are cylinders.

COR. 3. If a cone or cylinder be cut by a plane parallel to the base, the section is a circle.

COR.

COR. 4. Any plane touching the axis of a cone or cylinder, cuts every parallel section in its diameter.

PROPOSITION II.

Every sub-contrary section of an oblique cone or cylinder is a circle.

Let the cone or cylinder MBCN be cut by the plane MC, perpendicular to the base, touching the axis, and meeting the surface in the straight lines MB, NC, and the base in the diameter BC. Let it be cut by another plane DFG, perpendicular to the former, so that their line of common section DF may form with one of the lines MB, NC, the angle NFD, equal to the angle MBC, which BC forms with the other on the same side. The latter section DGF, which is called a sub-contrary section, is a circle, and DF its diameter. Fig. 41. 42.

For, let the parallel section HGI pass through any point K in DF. Because HGI is parallel to the base, it is perpendicular to MC, and therefore GK^a, its line of common section, with DGF, is perpendicular to the same^b. Thus, GK is at right angles to DF, and HI, and since HI is a diameter of the parallel section^c, the rectangle HKI = GK². But the triangles HDK, IKF, being similar (for the angles at K are equal, and

^a 4. c. 10. 6.

^b 8. 6.

^c 4. c. 1.

K k

the

the angle $NFD = MBC = DHK$ $DK : HK :: KI : KF$, and the rectangle $DKF = HKI$. Consequently the rectangle $DKF = GK^2$, and the section DGF a circle, having DF for a diameter.

COR. Every sub-contrary section of an oblique cylinder is equal to its base.

For the triangles HDK , IKF are isosceles.

PROPOSITION III.

Every oblique circular section of a cone or cylinder is a sub-contrary section.

Fig. 43. 44. Let $EHFG$ be an oblique circular section of the cone or cylinder MNP , the circle $EHFG$ is a sub-contrary section.

For, let any plane $ODRL$, parallel to the base, meet it in the line DKL , and let OKR be a diameter of the parallel section perpendicular to DKL . Also, let the plane MNP , of the axis, and the diameter OKR , meet the surface in the lines MN , PQ , the base in NP , and the circular section in EKF . And through any point C , in EF , let $AGBH$ be another parallel section, which the plane MNP intersects in the diameter ACB^a , and $EHFG$ in the chord GCH . Then, since the lines of common section of two parallel planes, with any third plane, are parallel ^b, AB is parallel

^a 4. c. 1.

^b 10. 6.

parallel to OR, and GH to DL. But OR is perpendicular to DL, therefore AB is perpendicular to GH^c. ^{c 11. 6.}
 And, since the diameter of a circle bisects the chord to which it is perpendicular, the chords GH, DL are bisected in the points C, K. Consequently the line EF, which bisects two parallel chords of the circular section EHFG, is a diameter of that circle, and perpendicular to these chords. Thus HCE is a right angle, and, since HCA is also a right angle, the line HC is perpendicular to the plane MNP^d. And consequently ^{d 2. 6.} the plane MNP is perpendicular to each of the planes AGBH, and EHFG, touching the line HC^e, and also ^{e 7. 6.} perpendicular to the base which is parallel to the former^f. ^{f 4. c. 10. 6.}

Again, because HC is perpendicular to the diameters EF and AB, the rectangle ECF = CH² = ACB, and EC : CA :: BC : CF: Therefore the triangles AEC, CBF are similar, and the angle CFB = EAC = ENP. Whence, since EHFG is perpendicular to the plane MNP, touching the axis, and MNP perpendicular to the base, EHFG is a sub-contrary section.

COR. No other than a parallel, and a subcontrary section of a cone, or cylinder, is a circle.

PROPOSITION IV.

Every section of a cone or cylinder, by a plane meeting the conic or cylindric surface on every side, that is neither a parallel nor a sub-contrary section, is an ellipse.

Let

Fig. 43-44. Let EHFG be a section of a cone or cylinder MNP, by a plane meeting the surface on every side, but neither a parallel nor a sub-contrary section, EHFG is an ellipse.

For, the same construction remaining, as in the preceding proposition, only let C be the middle of EF, it is shewn, in the same manner, that HC is perpendicular to AB, and that EF bisects GH and DL.

By similar triangles $EC : EK :: AC : OK$,

And $CF : KF :: CB : KR$.

Therefore $EC^2 : EKF :: CG^2 : DK^2$.

Consequently EHFG is an ellipse, of which EF and GH are two conjugate diameters.

PROPOSITION V.

If a cone be cut by a plane parallel to another, touching the conic surface, the section is a Parabola.

Fig. 45. Through any point B, in the circumference of the base BGC, let the tangent DE be drawn, the plane ADE, touching that line, and the vertex A, touches the conic surface in the straight line AB*. Let the cone be cut by the plane VGN, parallel to ADE, and meeting the base in GN; the section GVN is a parabola.

* cor. 4.

For,

FIG. 38.

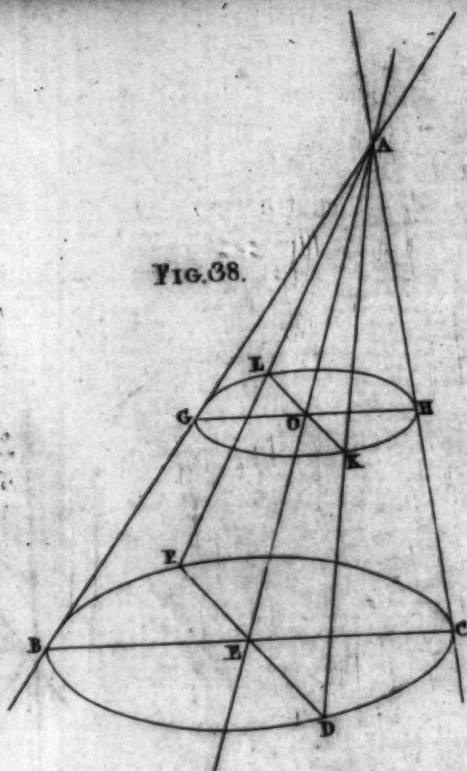


FIG. 39.

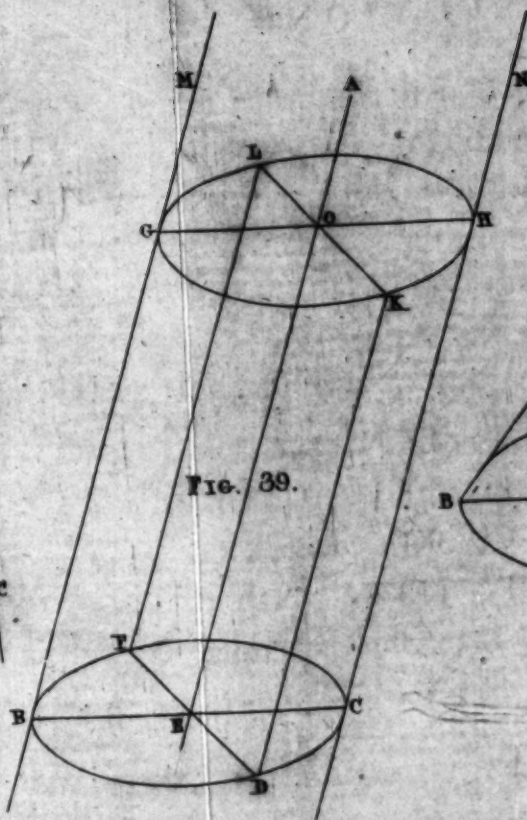


FIG. 40.

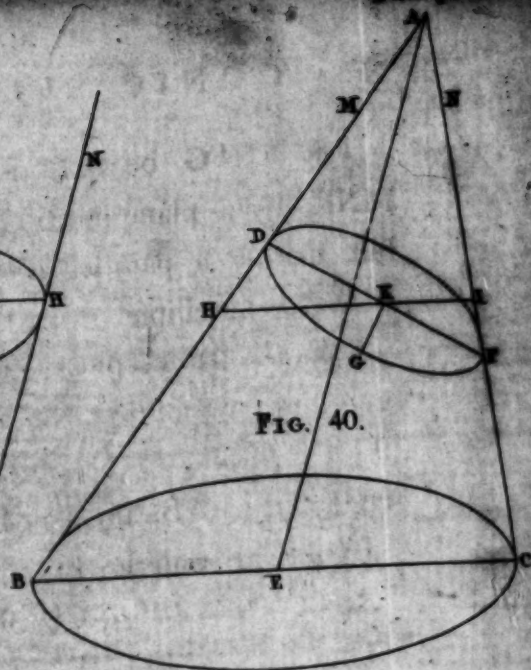


FIG. 41.

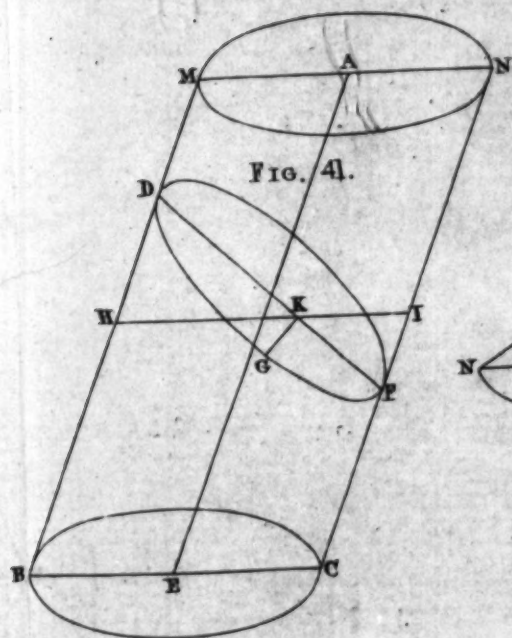


FIG. 42.

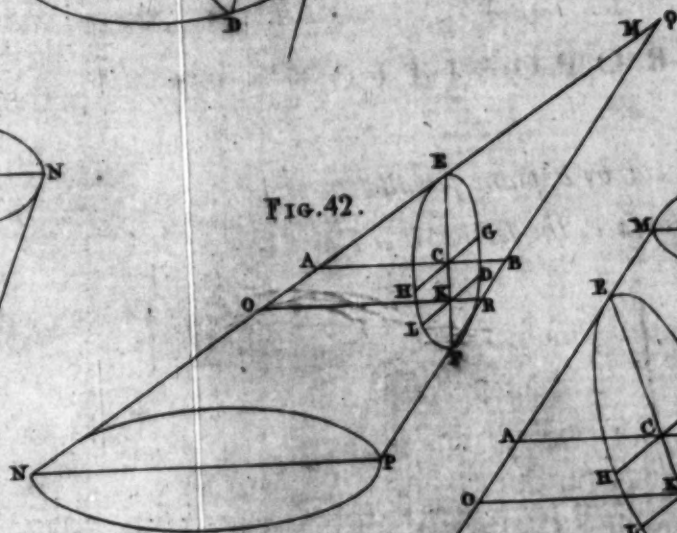
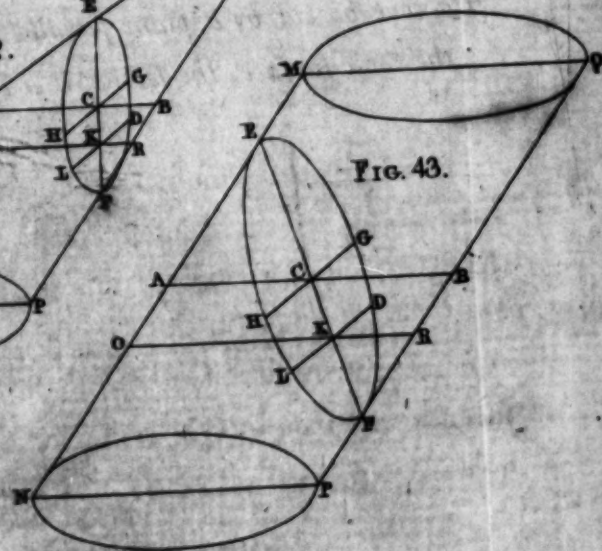


FIG. 43.



100

100

100

100

100

For, let ABC, the plane of AB and the axis, intersect GVN, in the line VK, and the base in the diameter BKC. Also, let MHL be a parallel section, through any point I in VK, intersecting ABC, in the diameter ML, and VKG in the straight line HI. Then, by reason of parallel planes, the straight line AB is parallel to VK, BD to GK, GK to HI, and BC to ML^b. Hence MK is a parallelogram; and, since DBC is a right angle, GK is perpendicular to BC, and HI to ML^c. Now, by similar triangles, VK : VI : KC : IL; but BK = ML. Therefore VK : VI :: BKC : MIL :: GK² : HI². Whence GVN is a parabola, of which VK is a diameter, and GK its semi-ordinate.

PROPOSITION VI.

Every other section of a cone is a hyperbola.

Let DFO be a section of the cone ABDC, different from any that has been mentioned, meeting the base in the line DO. Let DO be cut at right angles by the diameter BC, and let ABC, the plane of the axis, and BC, intersect the surface, and the plane FDO, in the straight lines AB, AFC, and FK. Then FK is not parallel to AB; otherwise, as DK is parallel to the tangent BG, the plane FDO would be parallel to the tangent plane AGB^a, contrary to the hypothesis; ^a 11. 6. let

Fig. 46.

let them meet, therefore, in the point E, which will be in the opposite surface, because, by hypothesis, the plane FDO does not meet the other on every side.

Also, let LMI be a parallel section, through any point N in FK, intersecting ABC in the diameter^b LI, and DKF, in the straight line MN. Then LI is parallel

to BC, and MN to DK^c, and therefore MN perpendicular to LI^a. Now, by similar triangles,

^a 11. 6.

$$EK : EN :: BK : LN$$

And $KF : NF :: KC : NI$

Therefore $EKF : ENF :: BKC : LNI :: DK^2 : MN^2$,

Whence DFO is a hyperbola, of which EF is a diameter, and DK, MN semiordinates.

PROPOSITION VII.

To find the radius of curvature at a given point, in a given conic section.

Fig. 47. CASE 1. Let D be the given point, in the given parabola DVL, at which it is required to find the radius of curvature.

From D, let the tangent DM, the perpendicular DR, and the diameter DF, be drawn. Also, through any point Q, in the curve near to the point D, let the circle DQO be described, to touch DM in D, and meet DF in P. Let PQ, QD be joined, and QN drawn

drawn parallel to MD, to meet DF in N. Then, because the angle $DPQ = MDQ = DQN$, the triangles DNQ , PQD , having a common angle at D, are equiangular.

Hence $PD : DQ :: DQ : DN$, and $PD \times DN = DQ^2$.

Also $PD^2 : PQ^2 :: DQ : QN^2$

Therefore $PD^2 \cdot PQ^2 :: PD \times DN : P \times DN$

Or $PD^2 : PQ^2 :: PD : P$, (P being the parameter of DF). Now, it is manifest that the nearer the point Q is to the point D, the nearer will the circumference of the circle be to a coincidence with the curve at that point; and, therefore, as no portion of these curves, however small, can be the same, the circumference of the circle will have approached the nearest possible to a coincidence with the curve at the point D, when the point Q falls upon it; in which case the last analogy, which holds universally, becomes $PD^2 : PD^2 :: PD : P$, where it is plain that PD is equal to P, the parameter of DF. Hence, if DP be taken equal to four times the distance of the given point from the directrix, and bisected by the perpendicular IR meeting DR, perpendicular to MD in R, then RD is the radius of curvature.

CASE II. In the ellipse, or hyperbola, let D be the given point, DM a tangent, DF, EG conjugate diameters, and DHO perpendicular to the two parallels DM, EG. Through any point Q, in the curve, near to the point D, let the circle DQO be described to touch

Fig. 48.

touch the line DM, and meet DF in P. Let PQ, QD be joined, and QN drawn parallel to DM, to meet DF in N. The triangles DNQ, PQD, are similar, $DN : DQ :: DQ : DP$, hence $DN : DP :: DN^2 : DQ^2$,

Or $DN : DP :: QN^2 : PQ^2$.

But $DF : P :: FN \times DN : QN^2$

Therefore $DF \times DN : P \times DP :: FN \times DN : PQ^2$.

And $DF : FN :: P \times DP : PQ^2$;

which analogy, when Q coincides with D, becomes $DF : DF :: P \times DP : DP^2$, where DP is equal to the parameter of DF, as before.

COR. 1. If a circle touch a tangent to a conic section, at the point of contact, on the same side, and cut off from the diameter which passes through that point, a segment, equal to its parameter, the conic section is of the same curvature with the circle at the point of contact.

COR. 2. If, from any point in the curve of an ellipse or hyperbola, a diameter be drawn, and a perpendicular be let fall upon its conjugate, the radius of curvature, at that point, will be a third proportional to the perpendicular and the semi-conjugate diameter.

For, since $DH : DC :: DP : DO$,

And $DC : EC :: EG : DP$.

$DH : EC :: EC : RD$.

PROPO.

PROPOSITION VIII.

If, from any point, in the curve of a conic section, an ordinate be applied to the axis, and likewise an ordinate to the diameter which passes through the other extremity of the first ordinate, the circle described through the extremities of the second, to touch the tangent to the curve, at the assumed point, will be of the same curvature with the conic section at that point.

From the point D, in the curve, let the ordinate DR be applied to the axis AB, and the ordinate DHL to the diameter RG; then the circle DPL described through D, L, to touch MD, the tangent to the curve at D, will be of the same curvature at that point. Fig. 49. 50.

For, let its circumference meet DF again in P, let PL be joined, and MD, RG intersect each other in E.

CASE I. In the parabola DAL, because the diameters are parallel, the angle DHE = PDL, and the angle MDP = DEH. But, since MD is a tangent to the circle, the angle MDP = PLD. Therefore the angle DEH = PLD, and the triangles EDH, DPL, equiangular. Hence $EH : HD :: LD : DP$. But the subtangent EH is bisected in R, and the ordinate LD in H. Therefore $RH : HD :: HD : DP$. Consequently DP is the parameter of RG, or DF and

L 1

DPL

DPL the circle of curvature corresponding to the point D.

Fig. 50.

CASE II. In the ellipse and hyperbola, because DR is an ordinate to the axis AB, the diameters DF and RG are equal, and the rectangle $ECH = CR^2 = CD^2$. Hence the triangles ECD and DCH are similar. Therefore the angle $PDL = DEH$, and the angle $PLD (= MDP) = DHE$; and consequently the triangles DPL and EDH are similar: Whence $DP : DL :: ED : EH :: ED \times CH : EH \times HC$. But $ED \times CH = CD \times DH$, and $EH \times HC = GH \times HR$. Therefore $DP : DL :: CD \times DH : GH \times HR$.

And $DP \times DH : DL \times DH :: CD \times DH : GH \times HR$.

By alterna-

tion $DP : CD :: 2DH^2 : GH \times HR$.

And by in-

version $DF : DP :: GH \times HR : DH^2$.

Consequently DP is the parameter of DF, and DPL the circle of curvature, corresponding to the point D.

A P P E N D I X.

PROP. I. Let there be a straight line, and a point without it, and let another point without it move in such a manner, that the ratio of its distances from these shall always be the same; that point shall de-

scribe

FIG. 44.

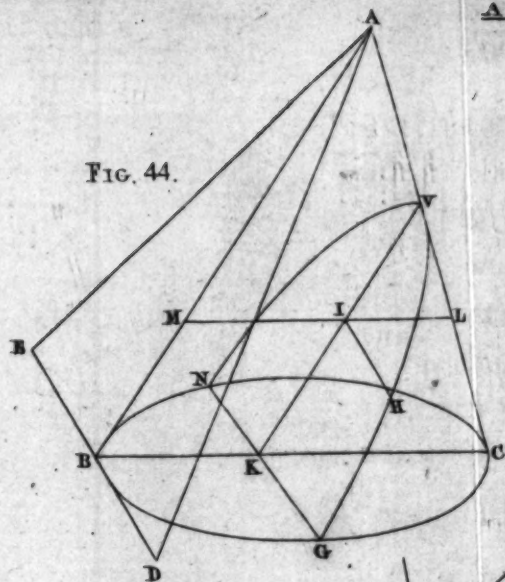


FIG. 45.

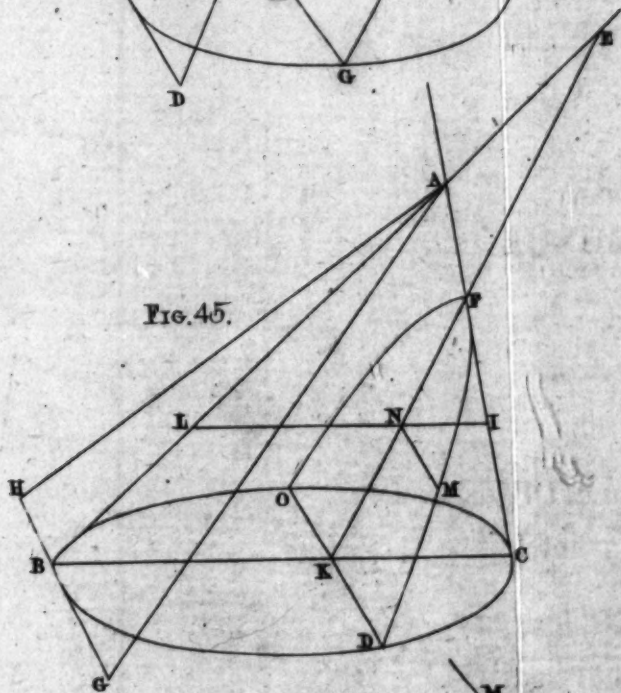


FIG. 47.

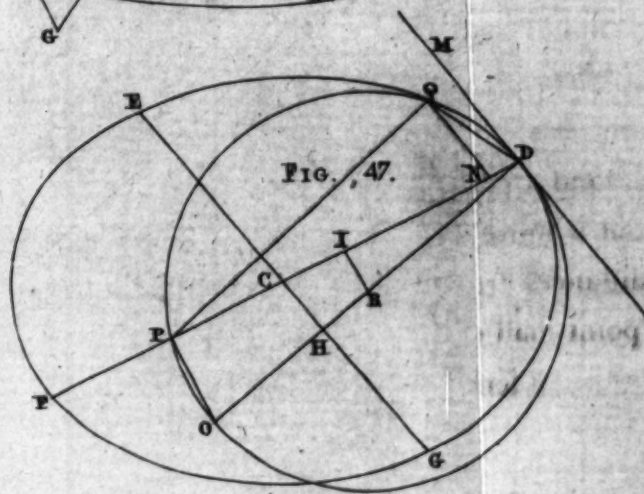


FIG. 46.

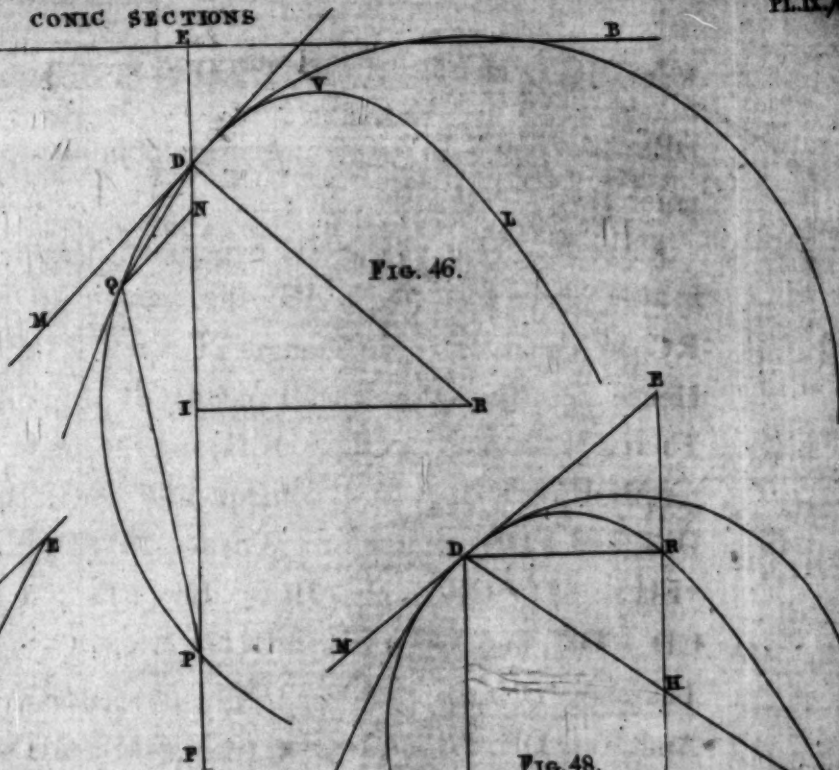


FIG. 48.

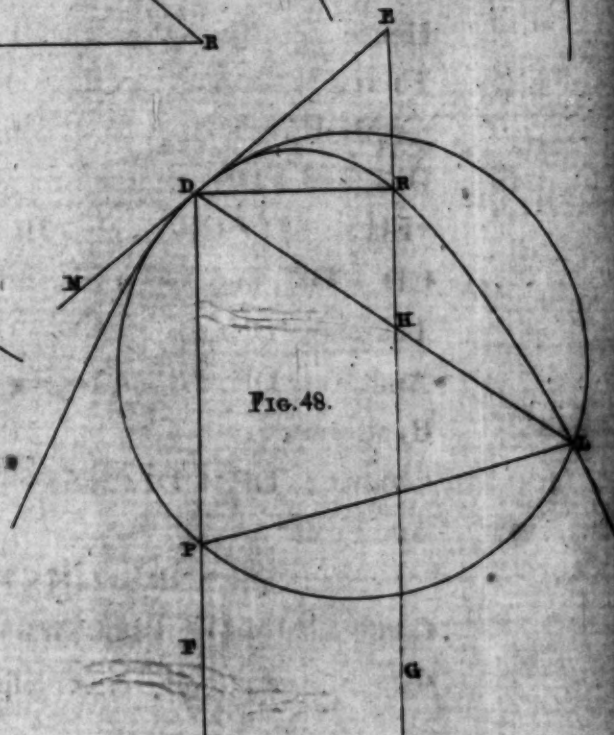
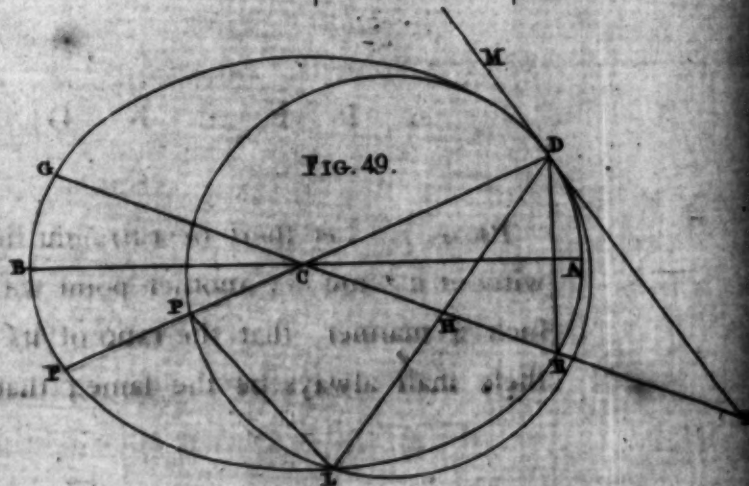


FIG. 49.



scribe an ellipse, parabola, or hyperbola, according as its distance from the other point is greater, equal, or less, than its distance from the straight line.

PROP. 2. Let a straight line touch the circumference of a circle; the point which moves equally distant from both describes a parabola. Let two circles touch one another, internally or externally; the point which moves equally distant from both their circumferences describes accordingly an ellipse or hyperbola.

PROP. 3. If two tangents to a conic section intersect one another, and, from any point in the one, a line be drawn parallel to the other, to intersect the curve, and the line joining the points of contact, its segments from that point will be continually proportional.

PROP. 4. If two parallel chords of a conic section be intersected by any third chord, the rectangles under the segments of the former will be to each other directly as the rectangles under the segments of the latter.

PROP. 5. If two chords of a conic section intersect, and be always parallel to two lines given in position, the rectangles under their segments will have to each other an invariable ratio.

PROP. 6. If, from the vertex of two tangents, a straight line be drawn through the focus, it will bisect

fect the angle formed by lines drawn through the focus, and the points of contact, or will be perpendicular to these lines, if they be in the same straight line.

PROP. 7. Any straight line, drawn from the vertex of two tangents, through the line joining their points of contact, to intersect the conic section in two points, is harmonically divided.

PROP. 8. A straight line drawn from the vertex of two secants, through the intersection of the lines which join the opposite points of section, to meet the conic section in two points, is harmonically divided.

PROP. 9. Any tangent intersecting two other tangents, and the line passing through their points of contact, is harmonically divided.

PROP. 10. If a straight line be drawn through the vertex of two tangents, parallel to the line joining their points of contact; any straight line drawn thro' the middle of the last line to meet the first, and intersect the conic section in two points, is harmonically divided.

PROP. 11. If two straight lines, not parallel, be inscribed in a conic section, and their extremes joined by four lines; the intersections of these and the vertices of the tangent triangles, erected upon the two chords, will be all four in one straight line.

PROP. 12. If a perpendicular to a tangent, from the point of contact, meet the axis of a conic section, and
from

from the point of concourse a perpendicular be let fall upon the line joining the focus and the point of contact, the segment of this line between the perpendicular and the tangent, is equal to half the parameter of the axis.

PROP. 13. Let there be any two chords in a conic section, and, from two extremes, which are not of the same chord, let two lines be drawn through any point in the curve, and from the other two extremes let two lines be drawn through any other point in the curve; the intersections of those lines which are not drawn from the same chord, nor meet each other in the curve, will be in a straight line, parallel to the chords, if the chords be parallel, or, if not, in the same straight line with their intersection.

MENSURATION.

MENSURATION.

B O O K I.

OF LINES AND ANGLES.

1. **E**VERY magnitude is measured by another of the same kind, called the measuring unit.

Thus, a line is measured by a line, an angle by an angle, a surface by a surface, and a solid by a solid.

2. Certain magnitudes being given, that is, their measures being determined by an actual application of the measuring unit, it is the business of mensuration to shew how the measures of others which depend on these may be obtained.

3. The first part of mensuration, which treats of lines and angles, being chiefly concerned about measuring the sides and angles of a triangle, is commonly named trigonometry.

4. Any

4. Any straight line of a determinate length may be assumed as the measuring unit of lines; but the 90th part of a right angle is the measuring unit of angles, and it is called an angle of one degree; and so every right angle is an angle of 90 degrees.

5. An angle at the center of a circle has the same ratio to four right angles, that the arch intercepted between its sides has to the whole circumference. Therefore the angle will be determined when it is known what part the arch is of the whole circumference. And hence the arch has been called the measure of the angle.

6. The circumference of every circle is supposed to be divided into 360 equal parts, called degrees, each degree into 60 minutes, each minute into 60 seconds, &c. marked thus, $25^{\circ} 14' 12'' 8'''$ (25 deg. 14 min. 12 sec. 8 thirds). Thus, a quadrant, or 4th part of the circumference, contains 90° , and an angle is of so many deg. min. &c. as are contained in its arch.

7. The complement of an arch, or angle, is its difference from a quadrant, or a right angle. Thus, the arches AB, BD, also the angles ACB, BCD, are the complements of each other. Fig. 1.

8. The supplement of an arch or angle is its difference from a semi-circle or two right angles. Thus, the arches AB, BL, also the angles ACB, BCL, are the supplements of each other.

9. The

9. The sine of an arch is a perpendicular falling from one extremity of the arch upon a diameter passing through the other. BE is the sine of AB.

10. The versed sine of an arch is that part of the diameter intercepted between the sine and the arch. AE is the versed sine of AB.

11. The tangent of an arch is a straight line touching it in one extremity, and limited by a line drawn from the center through the other extremity. AG is the tangent of AB.

12. The secant of an arch is the line which limits the tangent. CG is the secant of AB.

13. These are also said to be the sine, tangent, &c. of the angle ACB to the radius CA.

14. By the cosine, cotangent, &c. of an arch or angle, is meant the sine, tangent, &c. of its complement.

15. Hence, The chord of 60° , the sine of 90° , and the tang. of 45° , are each equal to the radius—An arch, and its supplement, have the same sine—The sine of an arch is half the chord of double the arch—The sine of an angle at the circumference is half the chord of the arch upon which it stands to the radius of that circle—The cosine of an arch is equal to that part of the diameter which is intercepted between the center and the sine.

16. The measures of three lines being given, the measure of the fourth proportional to them may be found

found by multiplying the 2d and 3d numbers together, and dividing their product by the first—The measures of two lines being given, that of a mean proportional between them, by multiplying the given numbers, and extracting the square root of their product—The measures of the two sides of a right angled triangle being given, that of the hypotenuse may be found, by adding the squares of the given numbers, and extracting the square root of the sum.

Let a, b, c , represent the measures of the hypo-
nuse, and the two sides of a right angled triangle, and
 m, n , those of the segments of the hypotenuse, then
 $a : b :: b : m$, and $a : c :: c : n$, hence $am = b^2$, $an = c^2$,
and $am + an = b^2 + c^2$, or $a^2 = b^2 + c^2$. Conse-
quently $a = \sqrt{b^2 + c^2}$, also $b = \sqrt{a^2 - c^2}$. Fig. 2.
2. c 12.6.

17. The solution of the three following problems is the object of the first part of mensuration.

PROBLEM 1. The measure of the diameter of a circle being given, to find that of its circumference.

PROB. 2. The measure of the radius being given, to find from thence the measures of the sine, tangent, secant, and versed sine, of every arch of a quadrant.

PROB. 3. The measures of three parts of a triangle being given, to find from these the measures of the remaining parts.

One method of measuring the circumference of a circle, and of constructing the trigonometrical canon, which is a table exhibiting the measures of the

M m

fine,

sine, tangent, secant, and versed sine of every arch of a quadrant, containing an exact number of minutes, is contained in the following propositions.

PROPOSITION I.

The supplemental chord of any arch is a mean proportional between the radius, and the sum of the diameter and supplemental chord of double the arch.

Fig. 3. The supplemental chord BE, of the arch AE, is a mean proportional between the radius BC, and the sum of the diameter AB, and supplemental chord BD, of the arch AD, the double of AE.

For, let BA be produced to F, so that $AF = BD$, and let ED, EC, EA, EF, be drawn. The triangles BDE, EAF, having the side $BD = AF$, $DE = EA$,
 * 1.c.10.3. and likewise the angle $BDE = FAE$, are equal in every respect. Thus, the angle at $F = DBE = EBA$, and the triangle FEB isosceles. Hence the triangles ECB, FEB are equiangular: Therefore $BC : BE :: BE : BF$,
 b 4. 5. or $BC : BE :: BE : AB + BD$.

COR. 1. The chord of any arch is a mean proportional between the radius and the difference of the diam. and supplemental chord of double the arch.

For, if BG be made equal to BD, and E, G joined, the triangles BGE, BDE are equal in every respect,
 and

and CAE, EAG similar. Hence $CA : AE :: AE : AG = AB - BD$.

COR. 2. If the radius $BC = 1$, and the supplemental chord BD be given in decimal parts of the radius, then,

$\sqrt{2 + BD} = BE$ the supplemental chord of half the arch AD .

$\sqrt{2 - BD} = AE$ the chord of half the arch AD .

PROPOSITION II.

The diameter of any circle is to its circumference as 1 to 3.1416 nearly.

Let ABC be a circle, in which AD is the side of a hexagon, and let the radius $AC (= AD) = 1$; then $BD = \sqrt{AB^2 - AD^2} = \sqrt{3}$. Thus, the supplemental chord of

$\frac{1}{2}$ of circumf. is	$\sqrt{3} = 1.7320508075 = a$
$\frac{1}{16}$	$\sqrt{2 + a} = 1.9318516525 = b$
$\frac{1}{64}$	$\sqrt{2 + b^2} = 1.9828897227 = c$ 2. c. 1.
$\frac{1}{256}$	$\sqrt{2 + c} = 1.9957178464 = d$
$\frac{1}{1024}$	$\sqrt{2 + d} = 1.9989291749 = m$
$\frac{1}{4096}$	$\sqrt{2 + m} = 1.9997322758 = n$
$\frac{1}{16384}$	$\sqrt{2 + n} = 1.9999330678 = p$
$\frac{1}{65536}$	$\sqrt{2 + p} = 1.9999832669 = q$
The chord of $\frac{1}{131072}$	$\sqrt{2 - q^2} = .004090612.$

The

1.c.21.3. The side of a regular polygon^b, therefore, of 1536 sides, inscribed in the circle, is .004090612. Now, let EF represent one of its sides, and GH one of the sides of the similar circumscribed polygon; then the perpendicular $CK = \sqrt{CE^2 - \frac{1}{4}EF^2} = \sqrt{2 + 9} = .999997908$; but $CK : CL :: EF : GH$, therefore .999997908 : 1 :: .004090612 : GH = .004090618, hence the perimeter of the inscr. polygon = .004090612 $\times$ 1536 = 6.283180032, the perimeter of the circumscribed polygon = .004090618 $\times$ 1536 = 6.283189248; consequently the circumference of the circle will be nearly = 6.283185, and the ratio of the diameter of any circle to its circumference 2 : 6.283185, or 1 : 3.1416 nearly.

^e 22. 5.

COR. 1. Let D, C represent the diameter and circumference of a circle, then $C = 3.1416 D$, and $D = .3183 C$.

COR. 2. To the radius 1, .01745 is the length of 1 degree, hence .01745 RN is the length of an arch of N degrees to the radius R.

COR. 3. An arch of a circle of the same length with the radius contains 57.295 degrees.

PROPO.

PROPOSITION III.

To find the sine and cosine of 1 minute to the radius 1.

It is evident that a very small arch, such as 1 minute, and its sine, will be very nearly equal; therefore

$$\frac{6.283185}{360 \times 60} = .0002908882 \text{ is the sine of 1 minute, and}$$

consequently $\sqrt{1 - .0002908882^2} = .9999999577$ is its cosine, or the sine of $89^\circ 59'$.

PROPOSITION IV.

Twice the rectangle under the sine of half the sum, and the cosine of half the difference of two arches, is equal to the rectangle under the radius and the sum of their sines.

Let AB, AD be the two arches, C the center, Fig. 5] AF a diameter, DL parallel to it, BG perpendicular to DL, CE perpendicular to the line joining B, D, and AH parallel to BD. Then the triangles BGD, CHA, which have the angles at G, H right, and the angles at D, A equal, by reason of parallel lines, are equiangular; therefore

BD

$BD : BG :: CA : CH$, and $2BI \times CH = CA \times BG$.
 Now, BI is the sine of BE , half the sum of the arches,
 CH is the cosine of AE , half their difference, and BG
 the sum of their sines. Whence the proposition is
 manifest.

COR. 1. If the sine of the mean of three equi differ-
 ent arches, the cosine of the common difference, and
 likewise the sine of one extreme be given in numbers,
 (the radius = 1) the sine of the other extreme may be
 found, by subtracting the latter of the given numbers
 from the product of the two former.

COR. 2. The sum of the sines of two arches, which
 together make 60° , is equal to the sine of an arch that
 is greater than 60° , by either of the two arches.

For, if $AB + AD = 60^\circ$, then $BD = CA$, and con-
 sequently $BG = CH = S, AN$, or $S, AM = S, \overline{AD + 60^\circ}$,
 or $S, \overline{AB + 60^\circ}$.

PROPOSITION V.

*To find the sines of all arches from 1 minute to 90 de-
 grees.*

Since the sine and cosine of 1 minute are given, let
 c denote the latter. Then, by Cor. 1. Prop. 4.

$$2c \times \text{fine } 1' - \text{fine } 0' = \text{fine } 2'.$$

$$2c \times \text{fine } 2' - \text{fine } 1' = \text{fine } 3'.$$

$$2c \times \text{fine } 3' - \text{fine } 2' = \text{fine } 4'.$$

$$2c \times \text{fine } 4' - \text{fine } 3' = \text{fine } 5'.$$

Thus, the fines of all arches of an exact number of minutes may be successively derived from one another. But the fines of those above 60° may be found by addition only, by Cor. 2. in this manner.

$$\text{Sine } 1' + \text{fine } 59^\circ 59' = \text{fine } 60^\circ 1'.$$

$$\text{Sine } 2' + \text{fine } 59^\circ 58' = \text{fine } 60^\circ 2'.$$

$$\text{Sine } 3' + \text{fine } 59^\circ 57' = \text{fine } 60^\circ 3'.$$

$$\text{Sine } 4' + \text{fine } 59^\circ 56' = \text{fine } 60^\circ 4'.$$

With respect to those arches which do not consist of an exact number of minutes, for the odd seconds in each, a proportional part of the difference between the fines of the next greater and less arches may be taken and added to the fine of the less, and that will give the fine of such an arch nearly.

PROPOSITION VI.

The fines of every arch of the quadrant being given, to find the tangents, secants, and versed fines.

Let

Fig. 6.

Let AB be any arch of the quadrant AD, BE its sine, CE its cosine, AE its versed sine, AG its tangent, and CG its secant. Then $AE = CA - CE$, and, by similar triangles, $CE : EB :: CA : AG$, also $CE : CB :: CA : CG$. Thus, the versed sine is equal to the difference between the radius and the cosine; the tangent is a fourth proportional to the cosine, the sine, and the radius; and the secant a third proportional to the cosine and the radius.

But the secants may be more easily found from the tangents. For the secant of any arch is equal to its tangent added to the tangent of half its complement. Let GF be made equal to GC, and CF joined: Then the angle $ACF = \frac{1}{2} G = \frac{1}{2} BCD$, therefore the arch $AH = \frac{1}{2} BD$, and AF the tangent of half the complement of AB.

On these principles may the trigonometrical canon be constructed; and, by means of it, the four cases of the third problem, both with respect to right angled and oblique angled triangles, may be resolved, the proportions for finding the unknown parts being first deduced from the following Theorems.

THEOREM

T H E O R E M I.

In any right angled triangle, the hypotenuse is to one of the sides as the radius (in the Tables) to the sine of the angle opposite to that side; also, one of the sides is to the other as the radius to the tangent of the angle opposite to the latter.

Let ABC be a right angled triangle, DE the radius Fig. 7.
of the tables, the angle EDF = BAC, FG its sine,
and EH its tangent. Then the triangle ABC is si-
milar to each of the triangles DGF, DEH, because
the angles at A, D are equal, and the angles at B, G,
E right. Hence $AC : CB :: DF : FG$, and $AB : BC$
 $:: DE : EH$, that is, $AC : CB :: R : S$, A, and
 $AB : BC :: R : T$, A.

COR. 1. To the hypotenuse as a radius, each side
is the sine of its opposite angle; and to one of the
sides, as a radius, the other side is the tangent of its
opposite angle, and the hypotenuse is the secant of
the same angle.

COR. 2. The sine, tangent, &c. of an arch or angle
have the same proportion to their radius, and to one
another, that the sine, tangent, &c. of a similar arch,
or equal angle, have to their radius, and to one ano-
ther.

Cases of Right-angled Triangles.

CASE 1. When the hypotenuse and one of the oblique angles are given.

CASE 2. When a side and one of the oblique angles are given.

CASE 3. When the hypotenuse and one of the sides are given.

CASE 4. When the two sides are given, to find the remaining parts of the triangle.

Proportions for solving these Cases.

CASE 1. Given the hypotenuse AC, and the angle A, to find the other parts of the triangle. As the oblique angles are the complements of one another, when the one is given or found, the other will also be known. Now, to find AB, BC,

$$R : S, A :: AC : CB$$

$$R : S, C :: AC : AB$$

$$\text{Or, } S', A : T, A :: AC : CB$$

$$S', A : R :: AC : AB$$

$$\text{Or, } S', C : R :: AC : CB$$

$$S', C : T, C :: AC : AB$$

The

The proportions for the other cases are to be stated in like manner. Those proportions which have radius for one of their terms are generally used in practice.

THEOREM II.

In any plane triangle, as one of the sides is to another, so is the sine of the angle opposite to the former to the sine of the angle opposite to the latter.

Let ABC be a triangle, then $AB : BC :: S, C : S, A$. Fig. 8.
About the center A, with the radius AB, let the semicircle DBE be described upon AC produced. Let the radius AF be drawn parallel to BC, and FG, BH perpendicular to the diameter DE. Then FG, BH are the sines of the angles (FAD) BCA, BAC, and the triangles FGA, BHC are similar : Hence $AF : BC :: FG : BH$, and therefore $AB : BC :: S, C : S, A$. * 2. c. 1. th.

THEOREM III.

In any plane triangle, as the sum of two unequal sides is to their difference, so is the tangent of half the sum of the opposite angles to the tangent of half their difference.

Let

Fig. 9.

Let ABC be a triangle, of which the side AC is greater than AB; then $AC + AB : AC - AB :: T, \frac{B+C}{2} : T, \frac{B-C}{2}$. About the center A, with the radius AB, let the semicircle DBE be described upon AC produced. Let DB, BE, and EF parallel to BC, be drawn. Then DC is the sum of AC, AB, and EC is their difference. Also DAB is the sum^a, and DEB half the sum^b of the opposite angles ABC, ACB; and BEF is half their difference^c. Now, to the same radius EB, DB is the tangent of DEB, and FB the tangent of BEF. And, because EF is parallel to BC, $DC : CE :: DB : BF$; hence $AC + AB : AC - AB :: T, \frac{B+C}{2} : T, \frac{B-C}{2}$.

T H E O R E M IV.

As half the base of any plane triangle is to half the sum of the two sides, so is half their difference to the distance of the middle of the base from the perpendicular.

Fig. 10.

Let ABC be a triangle, of which the base AC is bisected in E, and BD is the perpendicular upon AC.

^a 11. 2. Then, because $AC \times 2ED = \overline{AB+BC} \times \overline{AB-BC}$,
^b 2.c.8. 5. $AC : AB + BC :: AB - BC : 2ED$ ^b, and $\frac{1}{2} AC :$
 $\frac{AB + BC}{2} :: \frac{AB - BC}{2} : ED$.

COR.

COR. This fourth proportional ED added to half the base gives the greater segment; also $\frac{1}{2} AC \times ED =$

$$\frac{AB + BC}{2} \times \frac{AB - BC}{2}.$$

THEOREM V.

As the rectangle under any two sides of a plane triangle is to the rectangle under half the sum and half the difference of the base, and the difference of the two sides, so is the square of the radius to the square of the sine of half the included angle.

Let ABC be a triangle, having the side AB greater than BC. From AB let BD be cut off equal to BC, and from BC produced, BE equal to BA. Let AE, CD, and CG perpendicular to AE, be drawn. Also, let BHF be drawn to bisect the angle ABC, and it will bisect the bases AE, CD, at right angles. Now, from the right angled triangles ABF, DBH we have these proportions :

$$AB : AF :: R : S, \frac{1}{2} ABC.$$

$$BD : DH :: R : S, \frac{1}{2} ABC.$$

Consequently $AB \times BD : AF \times DH :: R^2 : S^2, \frac{1}{2} ABC^2.$ b 4 c. 9. 5.

Or $AB \times BC : AF \times FG :: R^2 : S^2, \frac{1}{2} ABC,$

There-

Therefore (Cor. Theo. 4.) $AB \times BC : \frac{AC + CE}{2}$
 $\times \frac{AC - CE}{2} :: R^2 : S, \frac{1}{2} ABC.$

COR. Hence $AC^2 = CE^2 + \frac{4AB \times BC \times S, \frac{1}{2} ABC}{R^2}$

Cases of Oblique angled Triangles.

CASE 1. When two angles and a side are given.

CASE 2. When two sides, and an angle opposite to one of them, are given.

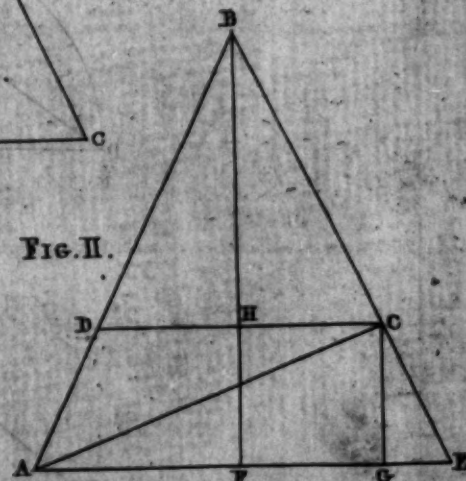
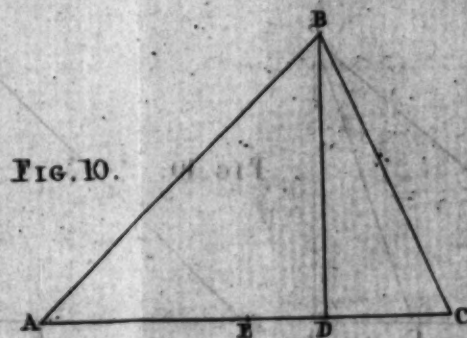
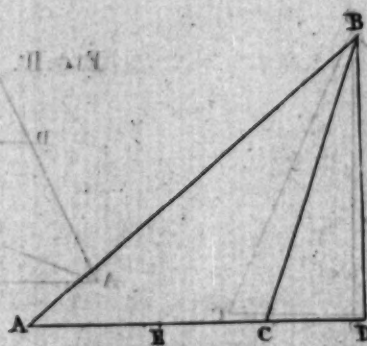
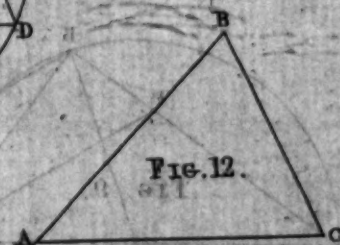
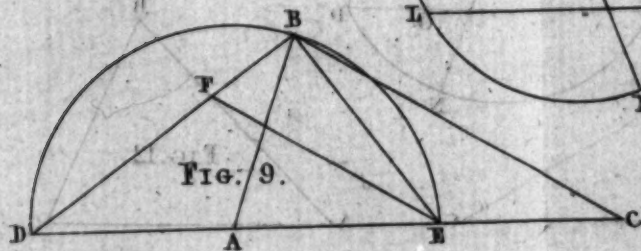
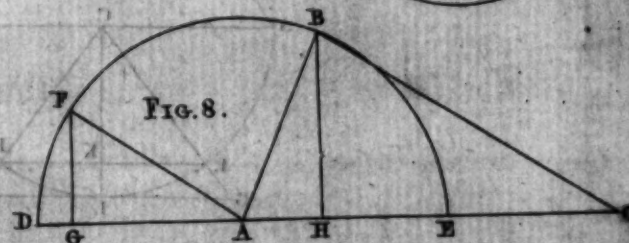
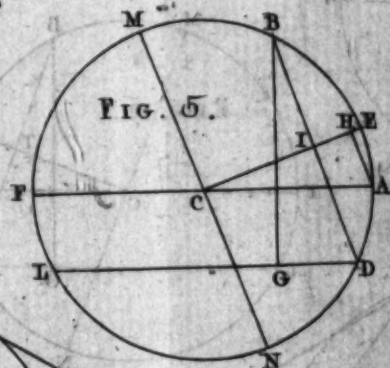
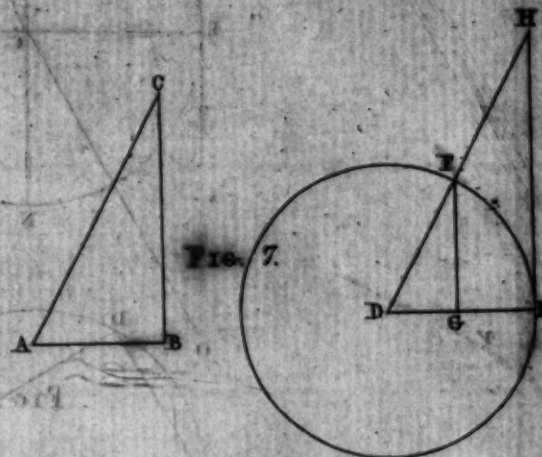
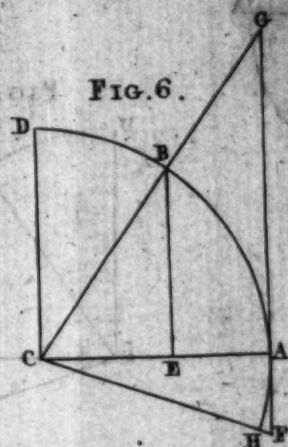
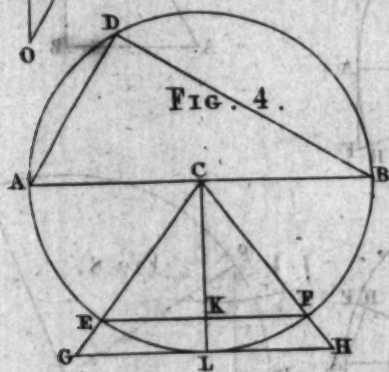
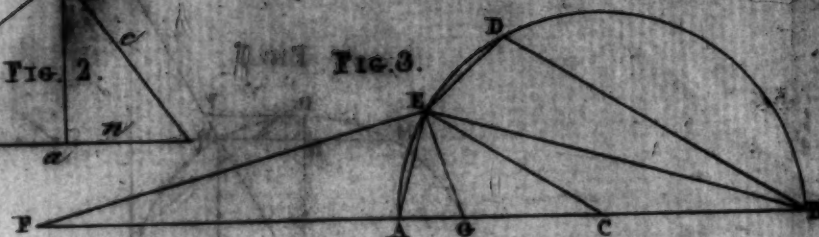
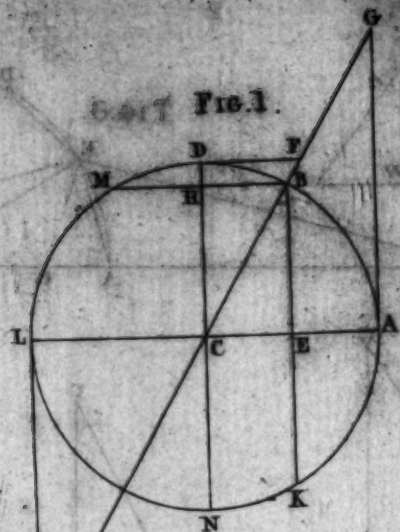
CASE 3. When two sides and the included angle are given.

CASE 4. When the three sides are given, to find the remaining parts of the triangle.

An angle of a triangle, and the sum of the other two, are the supplements of each other; therefore, when the one is given or found, the other will also be known. The proportions for the 1st and 2d cases are deduced from the 2d Theorem; but the data in the 2d case are not sufficient, when it is the less side that subtends the given angle.

In the 3d Case, half the difference of the required angles is found by the 3d Theorem, and that half difference added to the half sum, gives the greater angle; then the unknown side is found by the 1st Theorem.

The



The last Case will be resolved into the 3d Case of right angled triangles, if the segments of the base be found by the 4th Theorem, or it may be solved directly by the 5th Theorem.

Practical Solution of the Cases.

To avoid the tedious operations of Multiplication and division, in finding a fourth proportional, logarithms are generally used. These are a set of numbers, so adapted to the natural numbers, that the addition and subtraction of the former correspond to multiplication and division in the latter. However, in the proportions of the cases of right angled triangles, where radius is always one of the terms, the 4th term may be found with as much expedition by the natural as by the more artificial method of working.

Right angled Triangles.

CASE I. In the right angled triangle ABC, given Fig. 7. the hypotenuse AC 324, and the angle BAC $48^{\circ} 17'$, to find the other parts of the triangle.

The angle ACB = $90^{\circ} - 48^{\circ} 17' = 41^{\circ} 43'$. Then, to find AB, BC.

R :

$$R : S, A :: AC : CB$$

$$R : S, 48^\circ 17' :: 324 : CB$$

$$R : S, C :: AC : AB$$

$$R : S, 41^\circ 43' :: 324 : AB.$$

Solution by Natural Sines.

$$1 : .7464446 :: 324 : BC = 241.8480504.$$

$$1 : .6654475 :: 324 : AB = 215.60499.$$

Solution by Logarithmic Sines.

$$L. S, 48^\circ 17' \quad . \quad 9.8729976$$

$$L. 324 \quad . \quad 2.5105450$$

$$\text{Sum} \quad 12.3835426$$

$$L. R. \quad . \quad 10.0000000$$

$$L. BC \quad . \quad 2.3835426$$

$$L. S, 41^\circ 43' \quad . \quad 9.8231138$$

$$L. 324 \quad . \quad 2.5105450$$

$$\text{Sum} \quad 12.3336588$$

$$L. R. \quad . \quad 10.0000000$$

$$L. AB \quad . \quad 2.3336588$$

Whence $BC = 241.84805$, and $AB = 215.605$.

CASE

CASE 2. Given the side AB 125, and the angle A $51^{\circ} 19'$.

The angle C = $90^{\circ} - 51^{\circ} 19' = 38^{\circ} 41'$

$$R : \text{St. A} :: AB : AC$$

$$R : \text{St. } 51^{\circ} 19' :: 125 : AC$$

$$R : T, A :: AB : BC$$

$$R : T, 51^{\circ} 19' :: 125 : BC$$

Solution by Natural Secants and Tangents.

$$1 : 1.5964824 :: 125 : AC = 199.5603.$$

$$1 : 1.2489484 :: 125 : BC = 156.11855.$$

Solution by Logarithmic Secants and Tangents.

$$L. \text{ St } 51^{\circ} 19' \quad . \quad . \quad . \quad 10.2031641$$

$$L. 125 \quad . \quad . \quad . \quad 2.0969100$$

$$L. AC = 199.5603 \quad . \quad . \quad 2.3000741$$

$$L. T, 51^{\circ} 19' \quad . \quad . \quad . \quad 10.0965445$$

$$L. 125 \quad . \quad . \quad . \quad 2.0969100$$

$$L. BC = 156.1186 \quad . \quad . \quad 2.1934545$$

CASE 3. Given the hypotenuse AC 415, and the side AB 249.

$$AB : AC :: R : \text{St. A}$$

$$249 : 415 :: R : \text{St. A}$$

$$R : T, A :: AB : BC$$

$$R : T, A :: 249 : BC$$

$$249 : 415 :: 1 : 1.666 = \text{St. A } 53^\circ 7' 48''.3.$$

$$\text{L. } 415 + \text{L. R.} \quad . \quad 12.6180481$$

$$\text{L. } 249 \quad . \quad . \quad . \quad 2.3961993$$

$$\text{L. St. A } 53^\circ 7' 48''.3 \quad . \quad 10.2218488$$

$$1 : 1.333 :: 249 : \text{BC} = 332.$$

$$\text{L. T, } 53^\circ 7' 48'' \quad . \quad 10.1249384$$

$$\text{L. } 249 \quad . \quad . \quad . \quad 2.3961993$$

$$\text{L. BC} = 332 \quad . \quad . \quad 2.5211377$$

BC may be found without finding an angle.

$$\text{AC}^2 - \text{AB}^2 = \text{BC}^2.$$

$$172225 - 620001 = 110224.$$

$$\overline{\text{AC} + \text{AB}} \times \overline{\text{AC} - \text{AB}} = \overline{\text{BC}^2}$$

$$664 \times 166 = 110224.$$

$$\text{Therefore BC} = \sqrt{110224} = 332.$$

CASE 4. Given the side AB 53 and BC 67.

$$\text{AB} : \text{BC} :: \text{R} : \text{T, A}$$

$$53 : 67 :: \text{R} : \text{T, A}$$

$$\text{R} : \text{St, A} :: \text{AB} : \text{AC}$$

$$\text{R} : \text{St, A} :: 53 : \text{AC}$$

$$53 : 67 :: 1 : 1.26415094 = \text{T, A } 51^\circ 39' 15''.9.$$

$$\text{L. } 67 + \text{L. R} \quad . \quad 11.8260748$$

$$\text{L. } 53 \quad . \quad . \quad . \quad 1.7242759$$

$$\text{L. St. A } 51^\circ 39' 15''.9 \quad 10.1017989$$

$$1 : 1.$$

$$1 : 1.16118553 :: 53 : AC = 85.4283309$$

$$L. St. 51^{\circ} 39' 15'' . 10.2073260$$

$$L. 53 . 1.7242759$$

$$L. AC = 85.42834 . 1.9316019$$

Or, since $AC^2 = AB^2 + BC^2$, $AC = \sqrt{7298} = 85.428332$, &c.

Oblique angled Triangles.

CASE 1. Given the angle A $49^{\circ} 25'$, the angle C $63^{\circ} 48'$, and the side AB 275. Fig. 12.

$$\text{The angle B} = 180^{\circ} - 49^{\circ} 25' + 63^{\circ} 48' = 66^{\circ} 47'$$

$$S, C : S, A :: AB : BC$$

$$S, 63^{\circ} 48' : S, 49^{\circ} 25' :: 275 : BC$$

$$L. S. 49^{\circ} 25' + L. 275 . 12.3198379$$

$$L. S, 63^{\circ} 48' . 9.9529175$$

$$L. BC = 232.7665 : 2.3669204$$

$$S, C : S, B :: AB : AC$$

$$S, 63^{\circ} 48' : S, 66^{\circ} 47' :: 275 : AC$$

$$L. S, 66^{\circ} 47' + L. 275 . 12.4026580$$

$$L. S, 63^{\circ} 48' . 9.9529175$$

$$L. AC = 281.67 : 2.4497405$$

CASE 2:

CASE 2. Given the side AB 532, BC 358, and the angle C $107^{\circ} 40'$.

$$AB : BC :: S, C : S, A$$

$$532 : 358 :: S, 107^{\circ} 40' : S, A$$

$$L. S, 107^{\circ} 40', \text{ or } 72^{\circ} 20' + L. 358 \quad 12.5329022$$

$$L. 532 \quad \quad \quad 2.7259116$$

$$L. S, A \quad 39^{\circ} 52' 51''.7 \quad \quad \quad 9.8069906$$

$$S, C : S, B :: AB : AC$$

$$S. 107^{\circ} 40' : S, B :: 532 : AC$$

$$L. S. 32^{\circ} 27' 8''.3 + L. 532 \quad \quad 12.4555601$$

$$L. S. 72^{\circ} 20' \quad \quad \quad 9.9790192$$

$$L. AC = 299.6 \quad \quad \quad 2.4765409$$

CASE 3. Given the side AB 176, BC 133, and the included angle ABC 73° .

$$AB + BC : AB - BC :: T, \frac{A+C}{2} : T, \frac{C-A}{2}$$

$$309 : 43 :: T, 53^{\circ} 30' : T, \frac{C-A}{2}$$

$$L. T. 53^{\circ} 30' + L. 43 \quad \quad \quad 11.7642596$$

$$L. 309 \quad \quad \quad 2.4899585$$

$$L. T, \frac{C-A}{2} \quad 10^{\circ} 39' 2''.7 \quad \quad \quad 9.2743011$$

S, C

These Cases or Problems may be constructed geometrically, and the unknown parts measured by means of a scale of equal parts, and a line of chords.

A simple scale is a straight line divided into a certain number of equal parts, and having the first of these subdivided into 10 equal parts: So that, if the smaller divisions be taken for units, the larger will be tens; or, if the former be tens, the latter will be hundreds. Therefore, any number expressed by two digits may be taken from such a scale.

A diagonal scale is that which is described upon a simple one, in such a manner, as to exhibit the 10th parts of the smaller divisions, so that any number of three figures may be taken from it. The method of construction is evident from the figure.

A line of chords is a chord of 90° taken from any circle, and so divided, as to exhibit the chord of every arch of the quadrant, consisting of an exact number of degrees. The chord of 60° gives the radius of the circle from which it was taken.

E X A M P L E.

To construct the third case in oblique angled Triangles.

Fig. 13.

With the chord of 60° , as a radius, describe part of the circumference of a circle. Apply the chord of

73°

73° from *m* to *n*. From the center B draw the lines B*m*, B*n*. Upon the one set off 176 equal parts from B to A, and 133 upon the other, from B to C. Join A, C, and the triangle is constructed. Then AC, measured upon the same diagonal scale is 187, the angle A, measured by the same line of chords, is 43°, and the angle C 64°.

The measuring of Heights and Distances.

The instruments commonly used for measuring heights and distances are, a chain, a quadrant, a square, and a theodolite.

A chain is used for measuring those distances or lines which are to be given sides of triangles. The English chain is in length 4 poles, or 66 feet. It consists of 100 equal links, made of iron; each link, therefore, should be 7.92 inches long. Every ten links, from one end to the middle of the chain, is distinguished with a piece of brass.

A quadrant is used for determining vertical angles. It is made of brass or wood, the radius being of any convenient length, the circumference is divided into 90 equal parts, and these again subdivided, as far as the dimensions of the quadrant will admit. Also, a plummet is suspended by a thread from the center.

A square is used for finding the ratio of the sides of a right angled triangle. It is made of the same materials. Two of its sides are divided each into 100 equal parts. And a plummet hangs from the opposite angle.

A theodolite is used for measuring horizontal as well as vertical angles. It is a circle of brass divided into degrees, &c. having an index moveable about its center, and is furnished with sights.

P R O B L E M I.

To find the height of an accessible object standing upon level ground.

I. By the QUADRANT.

Fig. 14. Let any convenient distance BA be measured by the chain, in a direct line from the foot of the perpendicular BC, that falls from the top of the object. Then, standing at the point A, let the quadrant be held as represented in the figure, so that the eye at D may see the top of the object C, along the side of the quadrant DF. Now, if the plummet hang freely, the line FP will be perpendicular to the horizon, and therefore parallel to BC; hence the angles DFP, DCE are equal, and their complements GFP, CDE also

also equal. Thus GN, the arch of the quadrant that is remote from the eye, will give the number of degrees in the angle of elevation CDE. Whence, in the right angled triangle CED, the side DE (= AB,) and the angle CDE being given, we may find CE, to which EB, (= DA), the height of the eye above the ground being added, we get the whole height of the object.

If the angle of elevation be 45° , then $DE = EC$, that is, the distance measured is equal to the height of the object above the eye.

II. By the SQUARE.

Having measured AB, as before, hold the square to the eye at D, as in the figure. Then, the plummet hanging freely, the line FP cuts off from the square a small triangle similar to CDE. Therefore we shall have the ratio of DE to EC, and the former being given, the latter may be found by the rule of proportion. Let n represent the number of equal parts, which the plummet cuts off from the side DH or HG, towards the end D or G. Then, Fig. 15.

1. When the plummet cuts the side GH, remote from the eye, it is as $100 : n :: DE : EC$. Hence, if in this case, $DE = 100$, then $EC = n$.

2. When it passes through the opposite angle H, we have a ratio of equality, $DE = EC$.

P p

3. When

3. When it cuts the side DH contiguous to the eye, it is as $n : 100 :: DE : EC$.

P R O B L E M II.

To find the Height of an Inaccessible Object.

I. By the QUADRANT.

Fig. 16.

From any convenient station B, measure the distance BA in a direct line with the foot of the object, and at both stations A, B, take the angles of elevation DAC, DBC. The difference of these angles will give the angle ADB.

Then, in the triangle ABD, $S, ADB : S, A :: AB : BD$

And in the triangle BCD, $R : S, DBC :: BD : DC$

Whence $R \times S, ADB : S, A \times S, DBC :: AB : DC$,
that is, as the rectangle under the radius, and the sine
of the difference of the angles of elevation, is to the
rectangle under their sines, so is the distance mea-
sured to the height of the object above the eye.

ANOTHER SOLUTION.

In the right angled triangles DCB, DCA, to the
same radius DC, AC, CB are the cotangents of the
angles

angles of elevation, and their difference is AB; hence this proportion. As the difference of the natural cotangents of the angles of elevation is to the radius, so is the distance measured to the height of the object above the eye.

II. By the SQUARE.

At the station A find by the square the ratio of AC to CD, and, at the station B, the ratio of BC to CD; hence the ratio of AC to BC, and consequently that of AB to BC will be given, from which BC, and consequently CD may be found.

Let $AB = d$, $AD : CD :: m : n$, and $BC : CD ::$

$p : q$; then $CD = \frac{nqd}{mq - np}$. If, at both stations,

the plummet cut the side of the square remote from

the eye, $CD = \frac{nq}{q - n} \times \frac{d}{100}$. If the side conti-

guous to the eye, $CD = \frac{100d}{m - p}$. If the plummet

cut the opposite angles of the square at the first station

B, $CD = \frac{nd}{100 - n}$.

PROBLEM

P R O B L E M III.

To find the distance of a given place from an inaccessible object.

Fig. 17.

Let A be the inaccessible object, it is required to find its distance from the given station B. Measure any convenient distance BC, as the base of a triangle, whose vertex is at A. Then, the theodolite being placed at B, let the diameter be directed towards the station C, and the moveable index towards the object A, and the intercepted arch shews the number of degrees in the angle ABC. In like manner, let the angle BCA be measured, and the angle at A will be known. Then, $S, A : S, C :: CB : BA$.

P R O B L E M IV.

To find the distance between two inaccessible objects.

Fig. 18

Let a proper distance CD be measured as the base of two triangles, whose vertices are at the objects A, B. Then the angles at C and D being measured by the theodolite, we find, as in the last problem, the sides AD, DB: And, as the included angle ADB is

is given, the other angles of the triangle DBA may be found by the 3d theorem, and then the side AB by the 2d theorem.

A P P E N D I X.

Properties of Signs, Tangents, &c.

PROPOSITION 1. As the sum of the tangents of two arches is to their difference, so is the sine of the sum of those arches to the sine of their difference.

PROP. 2. Twice the rectangle, under the sine and cosine of any arch, is equal to the rectangle under the radius, and the sine of double the arch.

PROP. 3. As the sum of the sines of two arches is to their difference, so is the tangent of half their sum to the tangent of half their difference.

PROP. 4. As the sum of the cosines of two arches is to their difference, so is the cotangent of half their sum to the tangent of half their difference.

PROP. 5. Half the rectangle under the radius, and the sum of the sines of two arches, is equal to the rectangle under the sine of half their sum, and the cosine of half their difference.

PROP. 6. Half the rectangle under the radius, and the difference of the sines of two arches, is equal to the
the

the rectangle under the cosine of half their sum, and the sine of half their difference.

PROP. 7. Half the rectangle under the radius, and the versed sine of any arch, is equal to the square of the sine of half that arch.

PROP. 8. Half the rectangle under the radius, and the difference of the versed sines of two arches, is equal to the rectangle under the sines of half their sum and half their difference.

PROP. 9. As the sine of half the difference of two arches, which together make $\left. \begin{smallmatrix} 90 \\ 60 \end{smallmatrix} \right\}$ degrees is to the difference of their sines, so is 1 to $\left. \begin{smallmatrix} \sqrt{2} \\ \sqrt{3} \end{smallmatrix} \right\}$

PROP. 10. As the sine of half the sum of two arches is to the sine of half their difference, so is the sine of their sum to the difference of their sines.

PROP. 11. The difference between the tangent and cotangent of any arch is double the tangent of the difference between the arch and its complement.

PROP. 12. The sum of the tangent and cotangent of any arch is double the secant of the difference between the arch and its complement.

Trigonometrical

Trigonometrical Theorems.

THEOREM 1. In any right angled triangle, the sum of the hypotenuse and one side, is to the other side as the radius to the tangent of half the angle opposite to the latter.

THEOR. 2. The difference of the hypotenuse and one side is to the other side as the tangent of half the angle opposite to the latter is to the radius.

THEOR. 3. In any plane triangle, as the greater of the two sides is to the less, so is radius to the tangent of an angle; and radius is to the tangent of the excess of 45° , above this angle, as the tangent of half the sum of the angles at the base is to the tangent of half their difference.

THEOR. 4. As the base of any plane triangle is to the difference of the two sides, so is the sine of half the sum of the angles at the base to the sine of half their difference.

THEOR. 5. As the base is to the sum of the two sides, so is the cosine of half the sum of the angles at the base to the cosine of half their difference.

THEOR. 6. In any plane triangle, twice the rectangle under the two sides is to the excess of the sum of their squares, above the square of the base, as radius to the cosine of the vertical angle.

THEOR.

THEOR. 7. As the rectangle under the two sides is to the rectangle under half the perimeter, and its axis above the base, so is the square of the radius to the square of the cosine of half the vertical angle.

THEOR. 8. As the rectangle under half the perimeter, and its excess above the base, is to the rectangle under its excesses above the two sides, so is the square of the radius to the square of the tangent of half the vertical angle.

THEOR. 9. As half the perimeter is to its excess above the base, so is the cotangent of half either angle at the base to the tangent of half the other angle.

THEOR. 10. As the excess of half the perimeter above the less side is to its excess above the greater, so is the tangent of half the greater angle at the base to the tangent of half the less.

THEOR. 11. As the base is to the difference of its segments, so is the sine of the vertical angle to the sine of the difference of the angles at the base.

THEOR. 12. As radius to the sum of the cotangents of the angles at the base, so is the perpendicular to the perimeter.

Examples to the Cases

EXAMPLE 1. In a right angled triangle, given the hypotenuse 185, and one of the acute angles $32^{\circ} 40'$, to find the other parts of the triangle.

Ex.

Ex. 2. Given one of the sides 64, and the opposite acute angle $23^{\circ} 15'$.

Ex. 3. Given the hypotenuse 324, and one of the sides 265.

Ex. 4. Given one of the sides 78, and the other side 59.

Ex. 5. In an oblique angled triangle, given one of the angles $69^{\circ} 14'$, another $46^{\circ} 27'$, and the side between them 248, to find the remaining parts of the triangle.

Ex. 6. Given one of the angles 28° , another 114° , and the side between them 57.

Ex. 7. Given one of the sides 215, another 169, and the angle opposite to the former 74° .

Ex. 8. Given one of the sides 329, another 248, and the angle opposite to the latter 26° .

Ex. 9. Given one of the sides 79, another 67, and the included angle $85^{\circ} 16'$.

Ex. 10. Given one of the sides 241, another 173, and the included angle 103° .

Ex. 11. Given one of the sides 126, another 148, and the third 96.

Ex. 12. Given one of the sides 119, another 196, and the third 175.

Q d

Examples

Examples in measuring Heights and Distances.

Ex. 1. Required the height of a tower, whose angle of elevation, at the distance of 160 feet from the bottom, is $51^{\circ} 30'$.

Ex. 2. What is the height of a hill, whose angle of elevation at the bottom is 34° , and, 120 yards farther off, $29^{\circ} 15'$.

Ex. 3. To find the distance between two inaccessible objects, there are given the angles which it subtends, at two assumed stations, 60° and 66° , the distance between these stations 320 yards, and the other angles, at the same points, 43° and $44^{\circ} 30'$ respectively.

Ex. 4. To find the height of an object on the top of a hill, there are given the elevation of the hill 33° , and the elevation of the object at the same station 50° , also the elevation of the object, at another station, 80 yards, in a direct line from the former, 30° .

Ex. 5. From the top of a tower, whose height was 360 feet, the angles of depression of the top and bottom of an object, upon the same horizontal plane, were observed to be 41° and 54° respectively. The height of that object is required.

Ex. 6. From the summit of a hill, the angles of depression of the top and foot of a spire, upon the plane below, whose height was known to be 120 feet,

were

were observed to be 30° and 33° . Required the height of the hill?

Ex. 7. Two stations, at the distance of 60 yards, were taken on the side of a hill, from whence the angles of depression of the foot of a tower, on an opposite hill, were observed to be 70° and $80^\circ 30'$; also, at the upper station, the depression of the tower was found to be 18° , and the elevation of the tower 5° , and all the angles were taken in the same vertical plane. Required the height of the tower?

Ex. 8. On the top of a monument, whose height is 60 feet, stands a statue 12 feet high. At what distance from the monument may the statue be viewed, under an angle of 3° ? and what is the greatest angle under which it can be viewed?

Ex. 9. The elevation of a tower, at one station, is $20^\circ 45'$, at another station, 60 yards distant from the first, (but not in the same straight line with it, and the foot of the tower), $19^\circ 40'$, and, at a third station, 30 yards distant from each of the former $21^\circ 48'$. Required the height of the tower?

Ex. 10. If a statue, 12 feet high, on the top of a monument, appear to an observer under an angle of $4^\circ 46'$, and, at the same station, a part of the monument, of 20 feet from the top, subtend an angle of $8^\circ 30'$; What is the height of the tower, and its distance from the place of observation?

Ex.

Ex. 11. At three stations, in the same straight line with the foot of a tower, the angles of elevation are such, that the first is the double, and the second the complement of the third; also the distance of the first and second stations is 27, and the distance of the second and third 100 yards. Required the height of the tower?

Ex. 12. The elevations of a tower, at three several stations, (no two of which are in the same straight line with the foot of the tower), are $50^{\circ} 45'$, $58^{\circ} 15'$, and $46^{\circ} 45'$; also, the distance of the first and second stations is 24, the distance of the second and third 38, and the distance of the first and third 50 yards: What is the height of the tower?

MENSURATION.

MENSURATION.

B O O K II.

OF PLANE SURFACES.

A Straight line being assumed, as the measuring unit of lines, its square is assumed, as the measuring unit of surfaces. The number of measuring units which a surface contains is called its area. And the general problem, in this part of mensuration, is to find the area of any given plane figure.

The area of a rectangle may be found, by multiplying the numbers which express its length and breadth. Thus, if AB contain 5 times the measuring unit κ , and BC 3 times the same; then the rectangle ABCD will contain 15 times the measuring unit γ , which is the square of κ .

For,

Fig. 19.

For, let AE be equal to x , and EF parallel to AB ; then the ratio $AC : AF = AD : AE = 3 : 1$; therefore the rectangle AC is equal to 3 times AF ; but, since $AF : y = AB : x = 5 : 1$, AF is equal to 5 times y . Consequently AC is equal to 15 times y .

Hence the areas of all other rectilineal figures, also the areas of circles, ellipses, and parabolas, may be obtained by means of the following Theorems.

THEOREM I.

Fig. 20. *Every parallelogram is equal to the rectangle under its base and altitude.*

THEOREM II.

Every triangle is equal to half the rectangle under its base and perpendicular.

THEOREM III.

As radius is to the sine of one of the angles of a triangle, so is half the rectangle under the containing sides to the triangle.

THEOREM

THEOREM IV.

Every triangle is a mean proportional between the rectangle under half the perimeter and its excess, above one of the sides, and the rectangle under its excesses, above the two remaining sides.

Let the angles A and B, of the triangle ABC, be bisected by the lines AG, BG meeting in G. Also, let the exterior angle CBH be bisected by the line BK meeting AG in K. From the points G, K, let perpendiculars be drawn to each of the sides of the triangle, and let GC, CK be joined. Fig. 21.

Steps of the Demonstration.

1st, The perpendiculars GD, GE, GF, are equal; and the segments AD, AE; BD, BF; CE, CF equal.

2d, The perpendiculars KH, KM, KL, are equal: And the segments AH, AL; BH, BM; CM, CL equal.

3d, AH or AL is half the perimeter. Also $BF = CM$, and $BM = CF$.

4th,

4th, BH is the excess of half the perimeter above AB, DB its excess above AC, and AD its excess above BC.

5th, The rectangle under AH and DG is equal to the triangle ABC.

6th, From the similar triangles ADG, AHK; $AD : DG :: AH : HK$; therefore $AH \times AD : AH \times DG :: AH \times DG : HK \times DG$.

7th, The triangles GDB, BHK are similar; therefore $HK \times DG = DB \times BH$.

8th, Hence $AH \times AD : ABC :: ABC : DB \times BH$.

COR. Hence, if a, b, c represent the numbers which measure the three sides of a triangle, and s half their sum, the area of that triangle will be,

$$\sqrt{s \times s - a \times s - b \times s - c}$$

T H E O R E M V.

Fig. 22. Every quadrilateral figure is equal to half the rectangle, under one of its diagonals, and the sum of the perpendiculars falling upon it from the opposite angles.

T H E O R E M VI.

Fig. 23. Every quadrilateral figure, having two of its sides parallel, is equal to half the rectangle under the sum of the parallel sides, and the distance between them.

T H E O R E M

THEOREM VII.

In any quadrilateral figure, radius is to the sine of the angles formed by its diagonals, as half the rectangle under these diagonals to the quadrilateral. Fig. 24.

THEOREM VIII.

In any regular polygon, radius is to the tangent of half the angle of the polygon, as half its side to the radius of the inscribed circle; and the polygon itself is equal to half the rectangle under its perimeter, and the radius of the inscribed circle.

SCHOLIUM.

By this theorem, the areas of the following regular polygons (the side of each being 1) have been calculated, and put down in the Table opposite to their names. Hence it will be easy to find the area, when the side is expressed by any other number.

For, since similar polygons are to one another as the squares of their homologous sides, the area of any of these polygons may be found, by multiplying the

R r square

square of its side by the number standing opposite to its name in the Table.

N ^o . of Sides.	Names.	Areas.
3	Trigon.	0.433013
4	Tetragon.	1.000000
5	Pentagon.	1.720477
6	Hexagon.	2.598076
7	Heptagon.	3.633912
8	Octagon.	4.828427
9	Nonagon.	6.181824
10	Decagon.	7.694209
11	Undecagon.	9.365641
12	Duodecagon.	11.196152

T H E O R E M IX.

Every circle is to the square of its diameter as .7854 to 1 nearly.

For, the diameter of a circle being 1, its circumference is 3.1416 nearly, and every circle being equal to

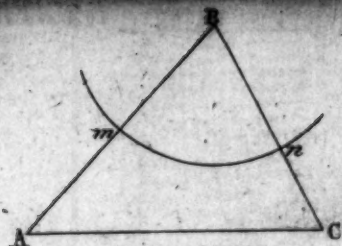


FIG. 14.

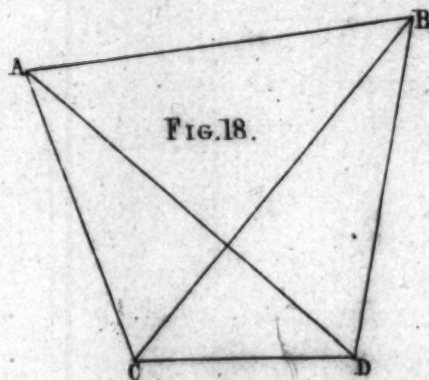
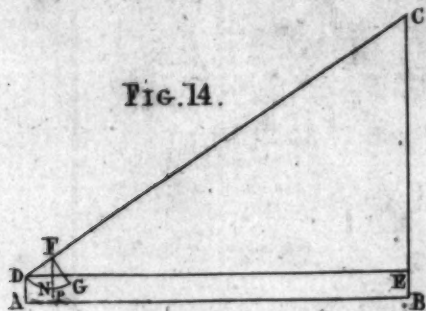


FIG. 18.

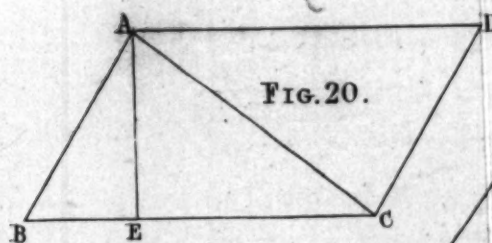


FIG. 20.

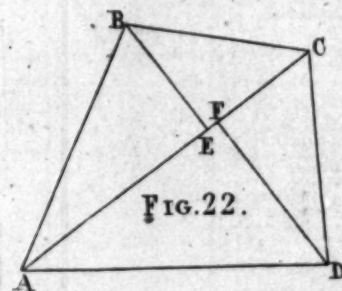


FIG. 22.

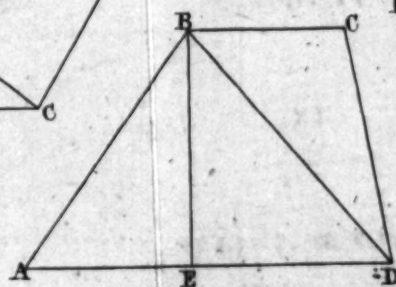


FIG. 23.

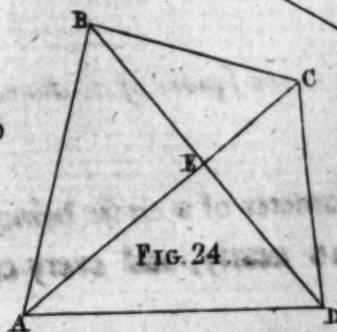


FIG. 24.

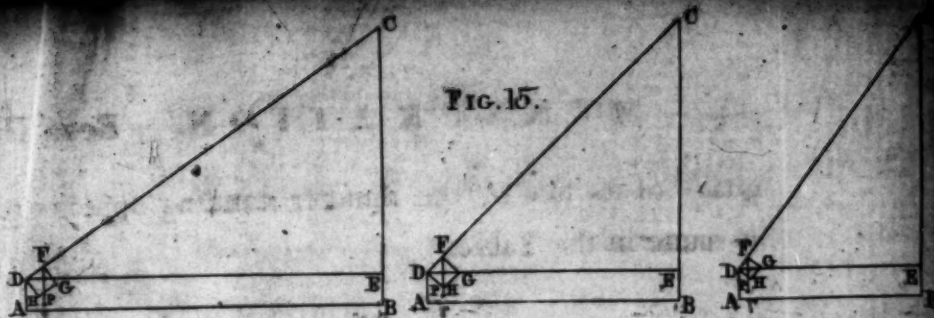


FIG. 15.

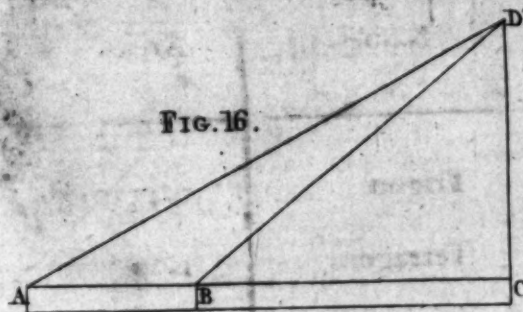


FIG. 16.

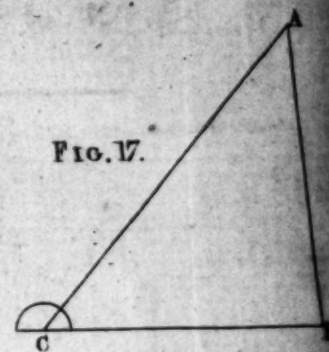


FIG. 17.

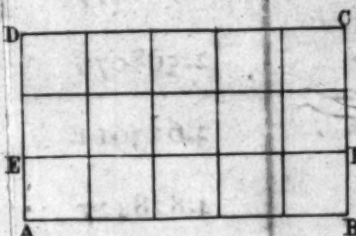


FIG. 19.

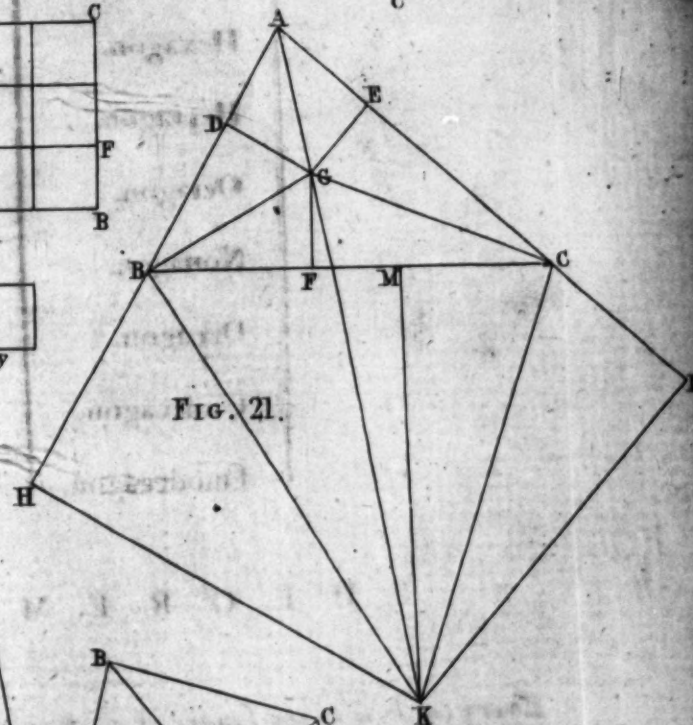
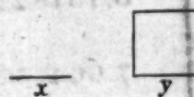


FIG. 21.

to half the rectangle under the radius, and a straight line equal to the circumference, its area is $= \frac{1}{2} \times \frac{1}{2} \times 3.1416 = .7854$. But circles are to one another as the squares of their diameters. Therefore $.7854 : 1$ is the ratio of any circle to the square of its diameter.

COR. 1. Let D, C, A represent the diameter, circumference, and area of a circle. Then $A = .7854D^2$, $A = .07958 C^2$, $D = \sqrt{1.2732 A}$, and $C = \sqrt{12.5664 A}$.

COR. 2. Let N be the number of degrees in the arch of a sector. Then $.01745 RN \times \frac{1}{2} R$, or $.008726 R^2 N$ is the area of the sector to the radius R.

THEOREM X.

Every ellipse is to the rectangle, under its transverse and conjugate axis, as .7854 to 1 nearly.

Let ACBD be an ellipse, of which AB is the transverse axis. Fig. 25.

From F, any point in the curve, let FG be perpendicular to AB, and meet the circle described upon it in E. Let FT, ET be the tangents at F, E, and thro' the point n, indefinitely near F, let mnp be drawn parallel to EG. Then,

The triangle EGT : FGT :: EG : FG

And mpT : npT :: EG : FG

Therefore Ep : Fp :: EG : FG :: AB : CD.

Hence

Hence every portion of the circle, between two parallel ordinates, is to the corresponding portion of the ellipse, in the constant ratio of AB to CD. Consequently the whole circle AHBK : ellipse ACBD :: AB : CD :: AB² : AB × CD. But AHBK : AB² :: .7854 : 1. Therefore ACBD : AB × CD :: .7854 : 1.

COR. 1. Ellipses are to one another as the rectangles under their axes.

COR. 2. Every ellipse is equal to a circle whose diameter is a mean proportional between the transverse and conjugate axis.

COR. 3. If a circle be described upon either axis of an ellipse, as that axis is to the other, so is the circle to the ellipse.

COR. 4. In the same ratio, are the segments, (and half segments), cut off by an ordinate to the common diameter. The parts intercepted between two ordinates (or semi-ordinates) to the common diameter. The segments formed by joining the extremities of these semi-ordinates. And the sectors formed by straight lines, drawn from any point in the common diameter, to the points where two perpendiculars to it meet the curves.

Thus, AB : CD = EBG : FBG = HOGE : COGF
= HE : CF = OLB : OMB = OLK : OMI.

THEOREM

T H E O R E M X I.

Every segment of a parabola is two thirds of its circumscribing parallelogram.

Let ABC be a segment of a parabola, cut off by Fig. 26.
the straight line AC. Let AC be bisected in F, the
tangent DBE drawn parallel to it, and B, F joined :
Then is BF a diameter, and AC its ordinate. Also,
let AD, CE be drawn parallel to BF. The parabola
ABC shall be two thirds of the parallelogram AE.

For, let AG, the tangent at A, meet BF, produced
in G, and through the points *o*, *t*, in the curve, or
tangent, equidistant from the point A, and indefinitely
near it, let *mn*, *xy*, *pq*, *rs*, be drawn parallel to AF,
AD. Then, because the parallelograms *An*, *Ap*, have
a common angle at A ; $An : Ap = AF : AD + Am : Aq$.
But $Am = oq$, and $oq : Aq = FG : AF$. There-
fore $An : Ap = AF : AD + FG : AF = FG : AD$.
And, since the subtangent FG is always bisected in
B, the vertex of the diameter, $FG = 2BF = 2AD$.
Whence $An : Ap = 2 : 1$. Thus, the parallelogram
An is the double of *Ap*, and the parallelogram *mncy*
the double of *pqrs*. But $otxn = mncy$, and $otsp = pqrs$.
Consequently *otxn* the portion of the parabola, be-
tween two indefinitely near semi-ordinates, is always
the

the double of *otsp*, the corresponding portion of the external space BDA. Therefore also the whole ABF is double the whole BDA. Whence ABF is $\frac{2}{3}$ of FD, and ABC $\frac{2}{3}$ of AE!

COR. 1. In like manner, may the quadrature of any curve, whose subtangent and absciss are in a constant ratio, be obtained.

Fig. 27.

COR. 2. Let AB, CD, EF, be three equidistant diameters, cut at right angles by the straight line BDF, the parabolic space ACEFB is equal to a rectangle under $AB + 4CD + EF$, and $\frac{1}{3} BD$.

For, let the diameters AB, EF meet the tangent at C in *m*, *n*. Then the quadrilateral

$$AF = \overline{AB + EF} \times BD.$$

$$\text{And } 2mF = 4CD \times BD.$$

$$\text{Consequently } (AF + 2mF =) 3ACEFB = \overline{AB + 4CD + EF} \times BD.$$

COR. 3. All parabolas, on the same, or equal bases, and having equal altitudes, are equal.

COR. 4. If two parabolas, on the same, or equal bases, be between the same parallels, any two parts, intercepted by lines parallel to the base, are equal.

COR. 5. If *C*, *c* be two parallel chords, in any parabola, and *D* the distance between them; then is

$$\frac{C^3 - c^3}{C^2 - c^2} \times \frac{1}{3} D, \text{ or } C + \frac{c^2}{C + c} \times \frac{1}{3} D, \text{ the expres-}$$

sion for the area of that part of the parabola which they intercept.

THEOREM

THEOREM XII.

Every sector of a parabola, formed by two straight lines, drawn from the focus to the curve, is equal to half the external space between perpendiculars falling from the extremities of the base upon the directrix.

From the extremities A, B, of the base of the sector AFB, let the perpendiculars AD, BE fall upon the directrix CD; the $AFB = \frac{1}{2} ADEB$.

Fig. 28.

For, let two points, p, q be assumed in the curve, or in the tangent AG, equidistant from the point A, and indefinitely near it. Let Fp, Fq be joined, mDn drawn parallel to AG, and the parallelogram $mnpq$ completed. Then, because the parallelogram mp , and the triangle Fpq stand upon the same base pq , and have equal altitudes, (for the tangent AG bisects FD at right angles), the former is double the latter. But $pqr = mp$; therefore $pqr = 2Fqp$, and consequently the whole external space ADEB double the whole sector AFB.

PROBLEM

P R O B L E M.

To find an approximation to the area of any curve different from a Parabola.

Fig. 29. Let AGN be any curve, different from a parabola, and BO any straight line, upon which, from the curve, there fall two perpendiculars, AB, NO, it is required to find an approximation to the area of the quadrilateral ABON. Let BO be divided into an even number of equal parts, by the perpendiculars CD, EF, GH, IK, LM, falling upon it from the curve, and let the number of these perpendiculars be such, that any two adjacent parts of the curve may nearly coincide with the curve of a parabola passing through their extremities, and having its axis parallel to the perpendiculars. Then the space

$$AEFB = \overline{AB + 4CD + EF} \times \frac{1}{3} BD \text{ nearly.}$$

$$EIKF = \overline{EF + 4GH + IK} \times \frac{1}{3} BD.$$

$$INOK = \overline{IK + 4LM + NO} \times \frac{1}{3} BD.$$

$$\text{Therefore } ABON = \overline{AB + 4CD + 2EF + 4GH + 2IK + 4LM + NO} \times \frac{1}{3} BD \text{ nearly.}$$

Hence, if F and L denote the first and last terms, in the series of numbers that measure the perpendiculars, E the sum of all the even terms, R the sum of all the rest, and D their common distance, we have
this

this general expression $\frac{F+L+4E+2R}{3} \times \frac{1}{3} D$ for an approximation to the area of the quadrilateral ABON, whatever be the curve AN.

E X A M P L E I.

Let it be required to find the area of the quadrant ABC, whereof the radius AC = 1.

Fig. 30.

Let AC be bisected by the perpendicular DE, and CD divided into 4 equal parts by the perpendiculars mn, pq, rs. Then $BCDE = \frac{BC + DE + 4mn + 4rs}{3} \times \frac{1}{3} CM$ nearly. Now, $BC = 1$, $DE = \frac{1}{2}\sqrt{3}$, $mn = \frac{1}{8}\sqrt{55}$, $rs = \frac{1}{8}\sqrt{63}$, $pq = \frac{1}{8}\sqrt{60}$, and $CM = \frac{1}{8}$. Thus, $BC + DE = 1 + \frac{1}{2}\sqrt{3} = 1.8660$
 $4mn + 4rs = \frac{1}{2}\sqrt{55} + \frac{1}{2}\sqrt{63} = 7.6767$
 $2pq = \frac{1}{2}\sqrt{15} = 1.9365$

$1.8660 + 7.6767 + 1.9365 = 11.4792$
 $11.4792 \times \frac{1}{24} = .4783 =$
 BCDE, from which, subtracting the triangle CDE $= .2165$

There remains the sector BCD $= .7618$

The triple of which is the quadrant ABC $= .7854$

Or the area of a circle, whose diameter is 1.

E X A M P L E

E X A M P L E II.

To find the area of the hyperbola FDM, of which the absciss $FM = 10$, semiordinate $MD = 12$, and semi-transverse $CF = 15$.

Fig. 31.

Let FM be divided into 5 equal parts, by the semiordinates HI, mn, pq, rs.

Then $HI = \frac{2}{5}$, $mn = \frac{6}{5} \sqrt{34}$, $pq = \frac{14}{5} \sqrt{6}$, and $rs = \frac{18}{5} \sqrt{19}$.

$$HI + MD = 16.8$$

$$4mn + 4rs = 68.8339$$

$$2pq = 17.6363$$

The figure HMD = $103.2702 \times \frac{1}{2} = 68.8468$ to which adding HIF = 6.4, considered as a portion of a parabola, we have 75.2468 for the area of the hyperbola.

E X A M P L E III.

Fig. 32.

Let AF, AE be the asymptotes, and C the vertex of an equilateral hyperbola; let CB be perpendicular to AE, and AB or BC = 1, required the area of the space BCDE, when BE = 1, and DE perpendicular

to

to AE. Let BE be divided into 6 equal parts, and the series of perpendiculars at the points of section will be $\frac{6}{7}, \frac{6}{8}, \frac{6}{9}, \frac{6}{10}, \frac{6}{11}$, $F + L = 1.5$, $4E = 8.27705$, $2R = 2.7$, and the figure BCDE $= 12.47705 \times \frac{1}{18} = .69316$.

OF SURVEYING.

The most useful instruments for surveying, are the chain and plane-table. A statute acre of land being 160 square poles, the chain is made 4 poles in length, that 10 square chains (or 100,000 square links) may be equal to an acre. The plane table is used for drawing a plan of the field, and taking such angles as are necessary to calculate its area. It is of a rectangular form, and is surrounded with a moveable frame, by means of which a sheet of paper may be fixed to its surface. It is furnished with an index, by which a line may be drawn on the paper, in the direction of any object in the field, and with scales of equal parts, by which such lines may be made proportional to the distances of the objects from the plane table, when measured by the chain; and its frame is divided into degrees, for observing angles.

PROBLEM

P R O B L E M I.

To measure a field with the chain.

Fig. 33.

Let $AmBCDq$ represent the field to be measured. Let it be resolved into the triangles AmB , ABD , BCD , AqD . Let all the sides of the large triangles ABD , BCD , and the perpendiculars of the small ones, AmB , AqD , from their vertices m , q , be measured by the chain, and the areas calculated: Their amount is the area of the whole.

But if, on account of the curvature of its sides, the field cannot be wholly resolved into triangles, then either a straight line may be drawn over the curve side, so that the parts cut off from the field, and those added to it, may be nearly equal, or, without going beyond the bounds of the field, the curvilinear spaces may be taken so small, that they may be considered as segments of a parabola, and measured accordingly.

P R O B L E M II.

To measure a field with the plane table.

Fig. 34.

Let the plane table be fixed at F , about the middle of the field $ABCDE$, and its distances FA , FB , FC ,

FC, &c. from the several corners of the field measured by the chain. Let the index be directed from any point assumed on the paper to the points A, B, C, &c. successively, and the lines Fa, Fb, Fc, &c. drawn in these directions. Let the angles which these lines contain be observed, and the lines themselves made proportional to the distances measured. Then, their extremities being joined, there will be formed a figure *abcde*, similar to that of the field; and the area of the field may be found by calculating the areas of the several triangles of which it consists.

P R O B L E M III.

To plan a field from a given base line.

Let two stations, A, B, be taken within the field, Fig. 35. but not in the same straight line, with any of its corners, and let their distance be measured. Then, the plane table being fixed at A, and the point *a* assumed on its surface directly above A, let its index be directed to B, and the straight line *ab* drawn along the side of it, to represent AB; also, let the index be directed from *a*, to an object at the corner *c*, and an indefinite straight line drawn in that direction; and so
of

of every other corner successively. Next, let the plane table be set at B, so that b may be directly over B, and ba in the same direction with BA, and let a straight line be drawn from b , in the direction BC. The intersection of that line with the former, it is evident, will determine the position of the point C, and the triangle abc , on the paper, will be similar to ABC in the field. In this manner are all the other angular points to be determined; and these being joined, there will be formed a representation of the field.

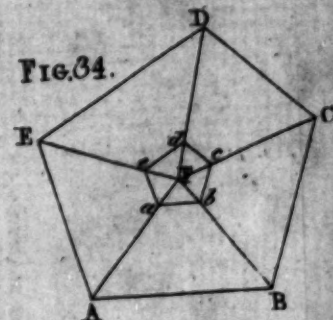
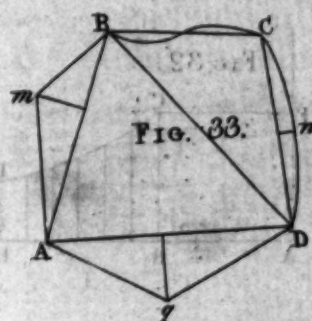
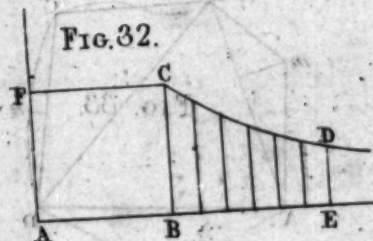
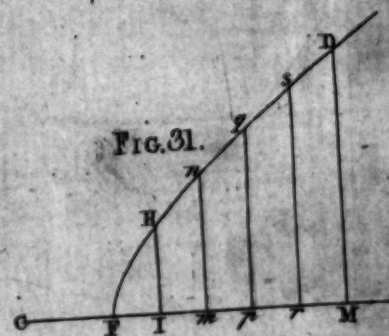
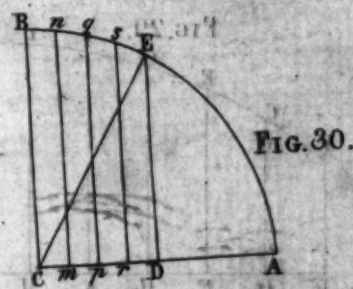
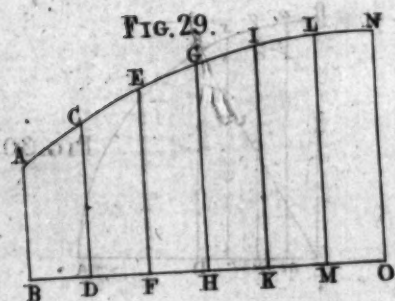
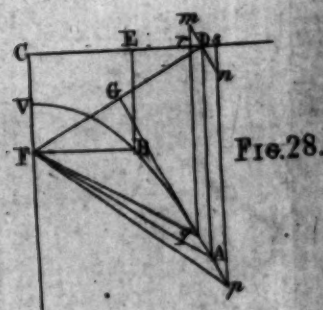
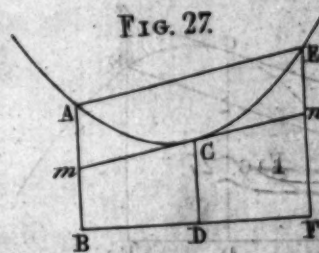
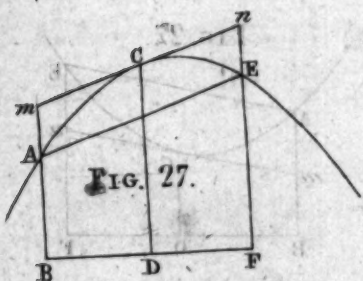
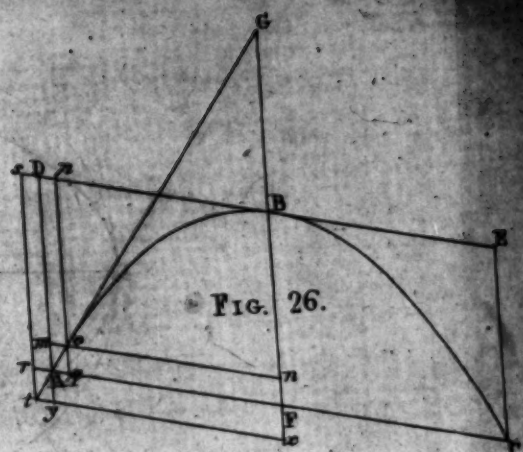
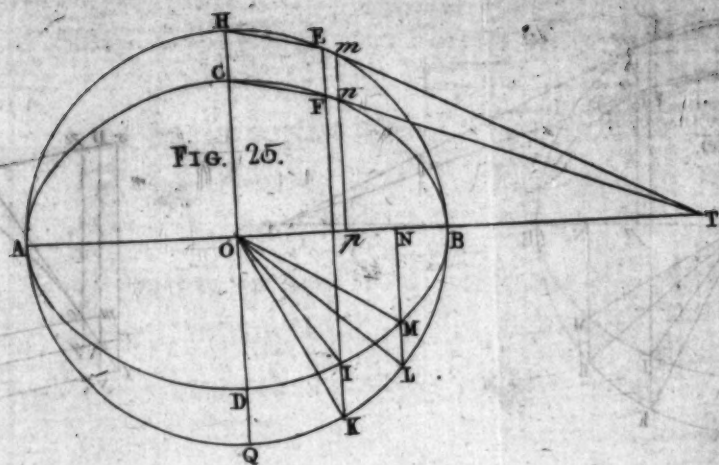
If the angles at both stations were observed, as the distance between them is given, the area of the field might be calculated from these data; but the operation is too tedious for practice; it is usual, therefore, to measure such lines in the figure that has been constructed, as will render the calculation easy.

A P P E N D I X.

T H E O R E M S.

THEOREM 1. Every right angled triangle is equal to the rectangle under half its perimeter, and the excess of half the perimeter above the hypotenuse.

THEOR.



MEASUREMENT

THEOREM 1. As radii is to the line of double
the acute angles of a right angled triangle
the square of half the hypotenuse to the square
of half the radius to the square of half the
hypotenuse of a triangle, for the rectangle under
half the perimeter, and its excess above the
square of half the perimeter.

THEOREM 2. Every quadrilateral figure inscribed in
a circle is a mean proportional between the excess of
the square of half the sum of two sides above the
square of half the difference of the other two, and the
excess of the square of half the sum of the latter
above the square of half the difference of the former.

THEOREM 3. Every quadrilateral figure inscribed in
a circle is a mean proportional between the rec-
tangle under the excesses of half the perimeter, above
two of the sides, and the rectangle under its excesses
above the other two sides.

NUMERICAL

THEOR. 2. As radius is to the sine of double one of the acute angles of a right angled triangle, so is the square of half the hypotenuse to the triangle.

THEOR. 3. As radius to the tangent of half the vertical angle of a triangle, so is the rectangle under half the perimeter, and its excess above the base to the triangle.

THEOR. 4. Every triangle is a mean proportional between the excess of the square of half the sum of the two sides above the square of half the base, and the excess of the square of half the base above the square of half the difference of the two sides.

THEOR. 5. Every quadrilateral figure inscribed in a circle is a mean proportional between the excess of the square of half the sum of two sides above the square of half the difference of the other two, and the excess of the square of half the sum of the latter above the square of half the difference of the former.

THEOR. 6. Every quadrilateral figure inscribed in a circle, is a mean proportional between the rectangle under the excesses of half the perimeter, above two of the sides, and the rectangle under its excesses, above the other two sides.

NUMERICAL

NUMERICAL PROBLEMS.

PROBLEM 1. Given two sides of a triangle 732, and 456, and the included angle 112° ; to find its area.

PROB. 2. To find the area of a triangle, whose sides are 273, 294, and 315.

PROB. 3. In the quadrangular field ABCD, given the side AB $8\frac{1}{2}$, BC $3\frac{1}{2}$, CD $9\frac{1}{2}$, and DA $10\frac{1}{2}$ chains, also ABC a right angle: Required the area?

PROB. 4. From the station F, within the five-sided field ABCDE, are measured FA 10.27, FB 9.35, FC 7.48, FD 7.85, and FE 7.15 chains; also the angle AFB 73° , BFC 75° , CFD 52° , and DFE 96° : Required the area?

PROB. 5. At two stations, A, B, within the field CDEFGH, distant 1250 links, were observed the following angles, ABC 137° , BAC $21^\circ 30'$, ABD $93^\circ 15'$, BAD 52° , ABE 25° , BAE 136° , EAF 70° , ABF $15^\circ 20'$, BAG 81° , ABG $70^\circ 30'$, BAH $26^\circ 40'$, and ABH $136^\circ 30'$: Required the area?

PROB. 6. The sides of a triangle are 252 and 240, and the base 156, also the sides of another triangle cut off from it next the vertex, are 60 and 47. Required the area of the remaining part of the triangle?

PROB.

PROB. 7. Given the sides of a triangle 169, 195, and 182 : Required the areas of the inscribed and circumscribing circles ?

PROB. 8. The sides of a quadrilateral inscribed in a circle, are 68, 43, 55, and 40 : Required the area ?

PROB. 9. The base of a triangle, and the perpendiculars falling from its extremities, upon the sides, are 123, 120, and 112.2, respectively : Required the sides and area of the triangle ?

PROB. 10. The vertical angle of a triangle, whose base is cut in extreme and mean proportion by the perpendicular, is 86° , and the sum of the including sides 112 : Required the sides and area ?

PROB. 11. The area of a triangle is 200, and the base, which is a mean proportional between the two sides, is 40 : Required these sides ?

PROB. 12. The two sides of a triangle are 640 and 360, and the base 800 : Required the line bisecting the vertical angle, and the areas of the triangles into which it divides the whole ?

PROB. 13. Required the sides and area of a triangle, of which the perpendicular is a mean proportional between the whole base and one of its segments, the vertical angle being 70° , and the sum of the including sides 100 ?

PROB. 14. A triangle whose sides are 123, 187, and 200, is divided into three triangles, which are to

T t

one

one another as 1, 2, 3, by straight lines drawn to the extremities of these sides, from the same point within the triangle, the areas and unknown sides of these triangles are required?

PROB. 15. The sides of a triangle are 300, 376, and 484, also the angles subtended by these sides, at a certain point within the triangle, are 104° , 112° , and 144° , respectively: Required the distances of that point from the angular points of the triangle.

PROB. 16. The sides of a quadrilateral inscribed in a circle are 195, 93, 149, 85: Required its area and diagonals, and likewise the area of the circle?

PROB. 17. In a right angled triangle, there is given the distance of one of the extremities of the hypotenuse from the center of the inscribed circle 4, and the produced part of the same, when continued, to meet the perpendicular 2; to find the hypotenuse and area?

PROB. 18. The sides of one triangle are 105, 252, 273, and the sides of another 396, 671, 715: Required the sides of a 3d triangle, similar to the 1st, and equal to the 2d?

PROB. 19. The angles at the base of a triangle are 70° and 80° , and the sides and base of another are 460, 510, and 830: Required the sides and area of a 3d, similar to the 1st, and described about the 2d, so that their bases are parallel?

PROB.

PROB. 20. Required the area of an ellipse, whereof the focal distance is 12, and the distance of the focus from the directrix 27 : Also, the areas of the segments of a parabola and hyperbola, cut off by an ordinate to the axis, the absciss being equal to the transverse axis of the ellipse, and the focal distance and parameter the same in each curve ?

MENSURATION.

MENSURATION.



BOOK III.

OF SOLIDS AND CURVE SURFACES.

A Cube, whose linear side is the measuring unit of lines, is assumed as the measuring unit of solids; and the number of measuring units which a solid contains is called its content or solidity.

The content of a rectangular parallelopipedon may be found by multiplying the three numbers which express its length, breadth, and thickness, continually together.

Fig. 36.

Thus, let ABCD-EFGH be a rectangular parallelopipedon, of which the linear side AB contains the measuring unit $\times$ 5 times, the side AD 3 times, and the

the side AE 4 times; then the rectangle AC contains y , or the square of x , 15 times, and the solid AG shall contain z , or the cube of x , 60 times.

For, let the solid AG be cut by the plane KLM, parallel to the base AC, and distant from it by AK, equal to x . Then, because $AC : y = AM : z$, and $AC = 15y$, $AM = 15Z$. But $AE : AK (= AH : AL) = AG : AM$, and $AE = 4AK$, consequently $AG = 4AM = 60z$.

Hence the solidities of prisms, pyramids, cylinders, cones, and spheres, also of parabolic, elliptic, and hyperbolic conoids, may be obtained by means of the following Theorems.

THEOREM I.

Every prism or cylinder is equal to a rectangular parallelepipedon of equal base and altitude.

THEOREM II.

Every pyramid or cone is one third of the prism or cylinder of the same base and altitude.

THEOREM III.

The frustum of any pyramid or cone is equal to three whole pyramids or cones of the same altitude with the frustum,

frustum, of which the greatest and least have their bases equal to the ends of the frustum, and the other is a mean proportional between them.

S C H O L I U M.

The demonstrations of these theorems are given in the sixth book of Geometry; and they may also be easily deduced from the following axiom: ‘Of two solids, between two parallel planes, if the sections, by a plane parallel to these, at any altitude, be in the same ratio with the bases, the solids themselves are to one another as their bases,’ upon which axiom the demonstrations of the remaining theorems depend.

T H E O R E M IV.

Every sphere is two thirds of its circumscribing cylinder.

Fig. 37.

Let the quadrant ACB, revolving around the radius AC, describe a hemisphere; the circumscribing square ADBC will at the same time describe a cylinder, and the triangle CAD a cone. Of these solids
the

the cylinder shall be equal to the sum of the other two.

For, through any point E, in AC, let EF be drawn parallel to BC, meeting the diagonal CD, the circumference AB, and the opposite side BD, in the points G, H, and F; and let C, H be joined. Because CEH is a right angled triangle, the circle described with the radius CH is equal to the sum of the circles described with the radii CE, EH. But CH = EF, and, since CA = AD, CE = EG. Therefore the circle described with the radius EF is equal to the sum of the circles described with the radii EG, EH; that is, of the three solids, the cylinder, cone, and hemisphere, which are between the same parallel planes, the section of the cylinder, at any altitude, is equal to the corresponding sections of the cone and hemisphere taken together. Consequently the cylinder is equal to the sum of the cone and hemisphere. But the cone is one third of the cylinder; therefore the hemisphere is two thirds of the same, and the whole sphere two thirds of the whole cylinder.

COR. 1. Hence $.5236 D^3$ expresses nearly the content of a sphere, whose diameter is D.

COR. 2. Every sphere is equal to a cone, of which the diameter of the base is equal to the diameter of the sphere, and the altitude double the same.

THEOREM

THEOREM V.

The frustum of a hemisphere is equal to the difference between a cylinder and a cone of the same altitude with the frustum, the former having the radius of the sphere, and the latter the altitude of the frustum for the radius of its base.

COR. If D be the diameter of the hemisphere, and A the altitude of the frustum, the solidity of the frustum is $.2618 A \times D^2 - A^2$.

THEOREM VI.

Every sector of a sphere, consisting of a segment and a cone, upon the same base, having its vertex in the center, is two thirds of a cylinder, having the same altitude with the segment, and the radius of the sphere for the radius of its base.

Fig. 37.

The spherical sector, described by AHC , is two thirds of the cylinder described by AF .

For, since the circle described by EF , is equal to the sum of the circles described by EH and EG , the cone

cone CEF is equal to the sum of the cones CEH and CEG. Let the double of the cone CEF be added to both, and the cone CEG be taken away, then the difference between the cylinder EB, and the cone CEG, that is, the frustum EHBC, is equal to double the cone CEF added to the cone CEH. Let the spherical segment AHE be added to both, then, the hemisphere ABC is equal to double the cone CEF, added to the spherical sector ACH, that is, two thirds of the cylinder AB equal to two thirds of the cylinder EB, added to the spherical sector AHC. Consequently the spherical sector AHC is two thirds of the cylinder AF.

COR. Every sector of a sphere is equal to a cone, of which the diameter of the base is equal to the diameter of the sphere, and the altitude double that of its segment.

THEOREM VII.

Every parabolic conoid is the half of its circumscribing cylinder.

Let there be a cylinder, and a parabolic conoid, described by the revolution of the rectangle ADCB, and the semi-parabola ACB, around the axis AB; the cylinder is double the conoid.

U u

For,

Fig. 38.

For, let BHD be the similar and equal curve, which passes through B, D, having the same axis AB, and the equal semi-ordinate AD. Through any point E, in AB, let EF be drawn parallel to BC, meeting the two curves, and the opposite side of the rectangle in G, H, F; and let P be the parameter of the axis.

Then, since $EG^2 = P \times AE$

And $EH^2 = P \times EB$

$$EG^2 + EH^2 = P \times AB = BC^2 = EF^2$$

Therefore, of the solids described by ADCB, ACB, ADB, between the same parallel planes, the section of the cylinder, at any altitude, is equal to the corresponding sections of the conoids taken together. Consequently, the cylinder is equal to both the conoids, and the double of either of them.

OTHERWISE.

Fig. 39.

Let GH be the semi-ordinate that bisects AB, and let EGF be parallel to AB. Then twice the square of GH is equal to the square of CB; and the sum of the squares of any two semi-ordinates, equally distant from GH, is equal to the square of CB. Hence the conoid AGH and frustum CGHB taken together, that is, the whole conoid ACB, is equal to the cylinder AF, which is half the cylinder AC.

THEOREM

THEOREM VIII.

The frustum of a parabolic conoid is equal to a cylinder of the same altitude, having its base equal to half the sum of the circular ends of the frustum.

THEOREM IX.

Every segment of a sphere is equal to the difference of a cone and cylinder, each of which has the altitude of the segment for the radius of its base, the former having also the altitude of the segment, but the latter the radius of the sphere for its altitude.

Let AHBC be a quadrant, C its center, ADBC the circumscribing square, AB, CD, its diagonals, and EF a straight line, drawn from any point E in AC, parallel to BC, and meeting AB, CD, AHB, BD, in the points M, G, H, F; and let the whole figure revolve around AC. The spherical segment, described by AHE, is equal to the difference between the cylinder and conic frustum, described by AF and AG^{th. 4.} But that difference is equal to the difference between the cylinder and cone described by AF and AME revolving upon EF.

For,

For, let the parabola AKB be described through A, B, having AE for its axis, and meeting EF in K. The radius AC, it is evident, is equal to the parameter of the axis; therefore $AC \times AE = EK^2$. But $2AC \times AE + EC^2 = AC^2 + AE^2$, $EM = AE$, $EG = EC$, and $EF = AC$. Therefore $2EK^2 + EG^2 = EF^2 + EM^2$. Hence the double of the conoid AKE, added to the conic frustum ADGE, is equal to the cylinder AF, added to the cone AME, since all these solids, described by their respective figures revolving around the common side AE, lie between the same parallel planes. But the double of the conoid AKE is equal to its circumscribing cylinder AK, and the cylinder AK is equal to the cylinder AF, having FE for its axis, because their bases and altitudes are reciprocally proportional. Therefore the cylinder AF, of which FE is the axis, added to the conic frustum ADGE is equal to the cylinder AF, of which AE is the axis, added to the cone AME, and the difference between the first and last of these solids, equal to the difference between the other two.

Consequently the spherical segment described by AHE, revolving upon AE, is equal to the difference of the cylinder and cone described by AF and AME, revolving upon EF.

THEOREM

THEOREM X.

Every elliptic conoid is two thirds of its circumscribing cylinder.

Let the elliptic quadrant ABC , and its circumscribing rectangle $AKBC$, revolving around the semi-axis AC , describe a conoid and cylinder, the former solid shall be two thirds of the latter. Fig. 41.

For, let the quadrant AGD be described about the center C , with the radius AC , let $AFDC$ be its circumscribing square, and let EG , parallel to CD , from any point E , in AC , meet the circle and ellipse in the points G, H . Then, because $EG : EH :: GD : CB$, the circles described by these lines are also proportional. Hence the hemisphere $AGDC$: conoid $AHBC ::$ cylinder $AFDC$: cylinder $AKBC$. Consequently the conoid $AHBC$ is two thirds of the cylinder $AKBC$.

COR. The segment of the conoid described by AHE is equal to the difference of the cylinder and cone described by AN and AME revolving around EN .

THEOREM

THEOREM XI.

Every spheroid, (described by the revolution of a semi-ellipse about either axis), is two thirds of its circumscribing cylinder.

THEOREM XII.

Every segment or frustum of the elliptic conoid is to the corresponding segment or frustum of the hemisphere as the square of the fixed axis of the ellipse is to the square of the other.

THEOREM XIII.

If an hyperbolic conoid, a parabolic conoid, and a cone, have the same base and altitude, the difference between the first and second of these solids, is to the difference between the second and third, as their common altitude is to the sum of that altitude, and the transverse axis of the generating hyperbola.

Fig. 42.

Let the three solids be described by the hyperbola AGCB, the parabola AFCB, and the triangle AHCB revolving

revolving upon AB; let DA be the transverse axis of the hyperbola, and let the straight line EHGF be drawn parallel to the semi-ordinate BC, from any point E in AB.

Then $BC^2 : EG^2 :: DB \times BA : DE \times EA$

And $EF^2 : BC^2 :: EA : BA$

Therefore $EF^2 : EG^2 :: DB : DE$

Hence $EF^2 - EG^2 : EF^2 :: BE : DB$

Again, because $EF^2 : BC^2 :: AE : AB$

And $BC^2 : EH^2 :: AB : AE$

Therefore $EF^2 : EH^2 :: AB : AE$

Hence $EF^2 : EF^2 - EH^2 :: AB : BE$

Consequently $EF^2 - EG^2 : EF^2 - EH^2 :: AB : DB$,
from which the proposition is manifest.

THEOREM XIV.

The frustum of a hyperbolic conoid is equal to the excess of the corresponding frustum of the asymptotic cone, above a cylinder of the same altitude, having the diameter of its base equal to the conjugate axis of the generating hyperbola.

Let AHD be a hyperbola, of which A is the vertex, C the center, and HK, DB semi-ordinates to the axis, meeting the asymptote CE in G, T; let AE be perpendicular to AC, and EL parallel to AB; and
let

Fig. 43.

let the whole figure revolve about the axis CA. The hyperbolic frustum HKBD is equal to the excess of the conic frustum GKBF, above the cylinder KL, because the excess of the square of GK, above the square of AE, is every where equal to the square of HKI.

COR. The whole hyperbolic conoid is equal to the excess of the circumscribing frustum of the asymptotic cone, above a cylinder of the same altitude, having the diameter of its base equal to the conjugate axis of the hyperbola.

T H E O R E M X V.

The curve surface of a right cone is equal to the sector of a circle, of which the radius is the hypotenuse of the generating triangle, and the arch equal to the circumference of the base of the cone.

T H E O R E M X V I.

The curve surface of the frustum of a right cone is equal to the rectangle under half the side of the frustum and a straight line equal to the sum of the circumferences of its ends.

T H E O R E M

FIG. 35.

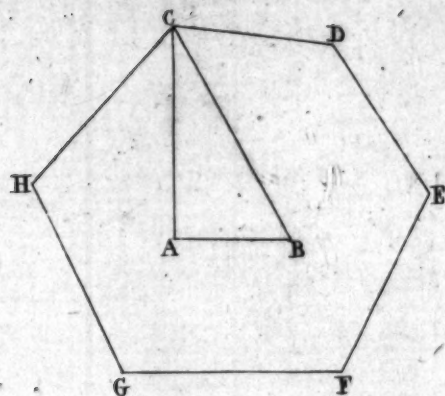


FIG. 36.

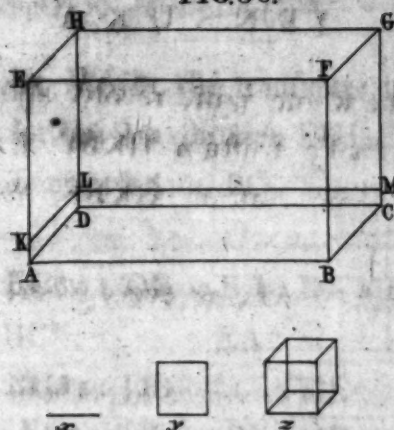


FIG. 37.

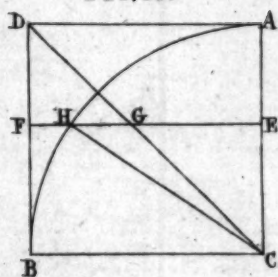


FIG. 38.

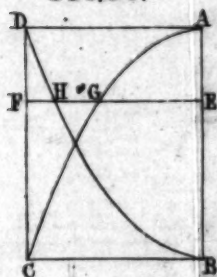


FIG. 39.

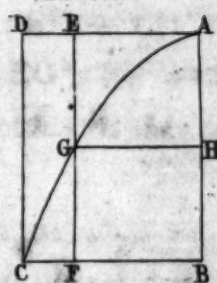


FIG. 40.

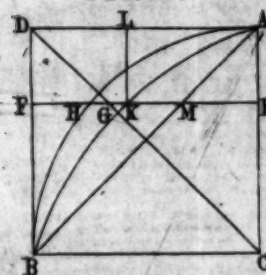


FIG. 41.

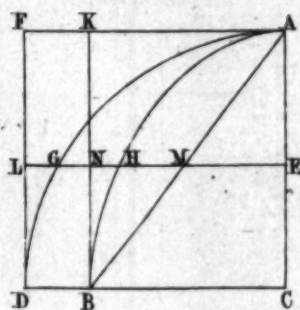


FIG. 42.

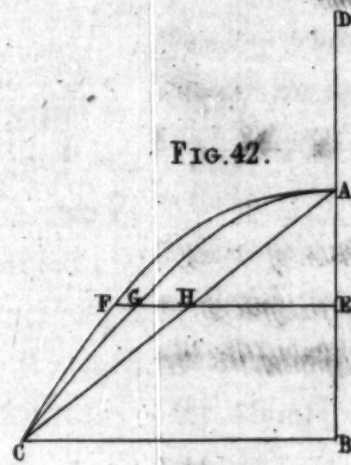
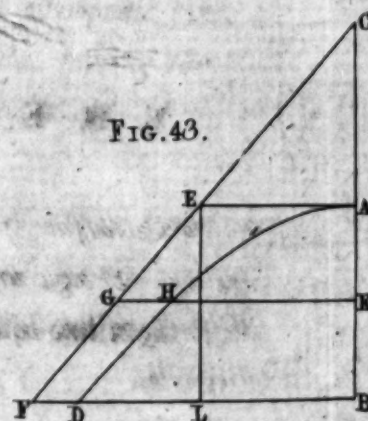


FIG. 43.



BOOK III. MISCELLANEOUS. 109

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T H E O R E M XVII.

The curve surface of a right cylinder is equal to the rectangle under its altitude, and a straight line equal to the circumference of its base.

T H E O R E M XVIII.

The surface of a sphere is equal to the rectangle under the diameter and circumference of the generating circle.

For, as the sphere may be conceived to be made up of an infinite number of cones, having a common vertex in the center, and their bases in the surface of the sphere, the whole sphere will be equal to one cone, whose base is the surface, and altitude the radius of the sphere. But the sphere is also equal to a cone, of which the diameter of the base is equal to the diameter of the sphere, and the altitude double the same. Wherefore these cones are equal to each other, and their bases and altitudes reciprocally proportional.

That is, $R : 2D :: \frac{1}{2} R \times C : S$

Hence $R \times C : D \times C :: R \times C : S$

Consequently $S = D \times C$.

X x

COR.

COR. The surface of a sphere is equal to a circle whose radius is the diameter of the sphere.

T H E O R E M XIX.

The curve surface of the segment of a sphere is equal to the rectangle under its altitude, and the circumference of the generating circle.

This theorem may be demonstrated after the same manner.

COR. The curve surface of the segment of a sphere is equal to a circle whose radius is the chord of the generating arch.

T H E O R E M XX.

The curve surface of the frustum of a sphere is equal to the rectangle under its altitude, and the circumference of the generating circle.

A P P E N D I X.

$$\text{Hence } R \times C : D \times C :: R \times C : S$$

$$\text{Consequently } S = D \times C.$$

A P P E N D I X.

NUMERICAL PROBLEMS.

PROBLEM 1. To find the surface and solidity of an upright triangular prism, of which the sides of the base are 36, 40, 68, and the altitude 80.

PROB. 2. To find the surface and solidity of an equilateral pyramid, of which the linear side is 60.

PROB. 3. Required the surface and solidity of a cylinder, inscribed in an upright triangular prism, whereof the sides of the base are 17, 25, 28, and altitude 30.

PROB. 4. Of a triangular pyramid, inscribed in a right cone, the sides of the base are 312, 96, 360, and the altitude 468 : Required the surface and solidity of the cone ?

PROB. 5. To find the surface and solidity of the frustum of a right cone, whose diameters are 18 and 8, and altitude 12.

PROB. 6. To find the surface and solidity of a sphere, whose diameter is 60, and of a segment thereof, whose altitude is 10.

PROB. 7. To find the solidity of the three conoids, the axis and diameter of the base in each being 45
and

and 30 respectively, 45 being also the transverse axis of the generating hyperbola.

PROB. 8. To find the surface and solidity of a sphere, whereof the circumference of the generating circle is 76.

PROB. 9. To find the surface and solidity of a cone, whereof the side is 30, and the circumference of the base 49.

PROB. 10. Required the altitudes of the three equal parts, into which the frustum of a cone, whose diameters are 36 and 8, and altitude 42, may be divided by sections parallel to the base.

PROB. 11. Required the weight of lead in a pipe 600 yards long, the diameter of the bore being $1\frac{1}{4}$ inches, the thickness of the metal $\frac{1}{4}$, and the weight of a cubic inch .40917 lb. Averdupois.

PROB. 12. A stone in the form of a rectangular parallelopipedon, measures 12 feet long, $7\frac{1}{2}$ feet broad, and 8 inches thick. A pound weight, of the same kind of stone, immersed in a cylindrical vessel of water 3.8 inches diameter, raises the fluid therein .85 of an inch. Required the weight of the stone?

SPHERICS.

S P H E R I C S.

B O O K I.

S P H E R I C A L G E O M E T R Y.

D E F I N I T I O N S.

1. **A** Sphere is a solid conceived to be generated by the revolution of a semicircle about its diameter.

2. The center of the semi-circle is equally distant from every point on the surface of the sphere, and is therefore called the center of the sphere.

3. Circles of the sphere, whose planes pass through the center, are called great circles, and all others small circles.

4. A straight line, drawn through the center of any circle of the sphere, perpendicular to its plane,
and

and limited on both sides, by the surface of the sphere, is called the axis of that circle.

5. The poles of a circle of the sphere are the extremities of its axis.

6. By the distance of two points on the surface of the sphere is meant an arch of a great circle intercepted between them.

7. By a spherical angle, or triangle, is always meant that which is formed on the surface of the sphere, by arches of great circles.

8. A quadrantal triangle is that of which one of the sides is a quadrant.

9. A lunary surface is a part of the surface of the sphere, contained by the halves of two great circles.

10. A segment of a sphere is a part cut off by a plane.

PROPOSITION I.

Every section of a sphere is a circle.

Fig. 1.

Let a plane cut the sphere AGH in any direction. If it pass through the center, the section is evidently a circle. But, if it do not pass through the center, let ABCD be the section, and from E, the center of the sphere, let EF be drawn perpendicular to its plane; also, let FA, FB, FC, be drawn in the plane, to meet the surface of the sphere. Then

EA'

EA, EB, EC, being joined, the right angled triangles EAF, EBF, ECF, have equal hypotenuses, EA, EB, EC, because they are radii of the sphere, and one side EF common to all; therefore their other sides FA, FB, FC, are also equal. Consequently ABCD is a circle, whose center is F.

COR. 1. Any two great circles cut one another in a diameter of the sphere, and therefore mutually bisect each other.

COR. 2. Only one great circle can pass through the same two points, in the surface of the sphere, that are not diametrically opposite.

COR. 3. Any side of a spherical triangle is less than a semicircle, and any two sides being produced, intersect again at the distance of a semicircle.

COR. 4. The two poles of any circle, its center, and the center of the sphere, are always in the same straight line, and that straight line is perpendicular to the plane of the circle.

COR. 5. And therefore, if a line or plane be perpendicular to a circle of the sphere, and pass through one of these points, it will pass through the other three: Or, if it pass through two of them, it will be perpendicular to the circle, and also pass through the remaining two.

COR. 6. Hence, two great circles, whose planes are perpendicular, pass through each other's poles; and conversely.

COR.

COR. 7. And, if one great circle pass through a pole of another, the latter will pass through the poles of the former.

COR. 8. All parallel circles have the same axis and the same poles ; and conversely.

P R O P O S I T I O N II.

Each pole of any circle of the sphere is equally distant on the surface from every point in its circumference.

Fig. 2.

Let ABCD be any circle of the sphere, whose center is F, and axis GFH : Its poles G, H are each of them equally distant from its circumference.

For, let the sphere be cut by planes passing through G, H, and let the sections, which will be great circles, because the line GH passes through the center of the sphere, meet ABCD in the lines of common section FA, FB, FC. Then GA, GB, GC, being joined, the right angled triangles GFA, GFB, GFC, have the sides FA, FB, FC, equal, because they are radii of the same circle, and one side GF common to all ; therefore the hypotenuses GA, GB, GC, and consequently the arches which they subtend, are likewise equal.

COR. 1. The pole of a great circle is at the distance of a quadrant from its circumference.

COR.

COR. 2. Hence any plane passing through the center of the sphere, divides it into two equal parts, which are therefore called hemispheres.

COR. 3. If a point in the surface of the sphere be at the distance of a quadrant from other two points not diametrically opposite, it will be the pole of the great circle passing through them.

COR. 4. The radius of a small circle is the sine of its distance from either pole to the radius of the sphere, or the cosine of its distance from the parallel great circle.

COR. 5. Hence, those small circles, whose planes are equally distant from the center, are equal ; and conversely ; and of two circles unequally distant, that which is nearer the center is the greater ; and conversely.

COR. 6. Parallel circles intercept equal arches on those great circles which pass through their poles.

PROPOSITION III.

The intercepted arch of a great circle, whose pole is the angular point, is the measure of a spherical angle.

Let ABC be a spherical angle, of which the angular point B is the pole of the great circle ACD. Then is the intercepted arch AC the measure of ABC.

Fig. 3.

Y y

For,

For, let the tangents MB, NB, and the radii of the sphere EA, EB, EC, be drawn. The angle MBN is the same with the spherical angle ABC; but MBN is equal to AEC; because, since AB, BC are quadrants^a, and BEA, BEC right angles, MB, BN are parallel to AE, EC. Wherefore the spherical angle ABC is equal to AEC, the measure of which is the arch AC to the radius of the sphere.

^a 1. c. 2.

COR. 1. The circumferences of two great circles cut each other at right angles, when their planes are perpendicular; and conversely.

COR. 2. At the point of intersection of two great circles, the opposite angles are equal, two adjacent angles are together equal to two right angles, and each angle is equal to its opposite one, at the other point of intersection.

COR. 3. The distance of the adjacent poles of two great circles is the measure of their inclination, or of the spherical angle.

For, since AB, BC pass through the poles of ACD, ACD passes through the poles of AB, BC^b; let P be the pole of AB and Q, the adjacent pole of BC. Then AP, CQ are quadrants, and AC = PQ.

^b 7. c. 1.

COR. 4. The intercepted arch of any circle, whose pole is the angular point, is the measure of the spherical angle to the radius of that circle.

COR. 5. Two great circles, which pass through the poles of parallel circles, intercept similar arches.

P R O P O.

PROPOSITION IV.

If a plane be perpendicular to the diameter of a sphere, at one of its extremities, it touches the sphere.

Let the plane CD be perpendicular to AB, the diameter of the sphere ABG, at its extremity B; then CD touches the sphere in that point. Fig. 4.

For, let F be any other point in CD, and EF, FB be drawn. The angle EBF is, by hypothesis, a right angle. Hence EF is greater than EB, and consequently F a point without the sphere. Thus the plane CD meets the sphere only in the point B, and therefore touches it.

COR. 1. A sphere and a plane can touch one another only in one point.

COR. 2. If a plane touch a sphere, the radius at the point of contact is perpendicular to it.

COR. 3. If a plane touch a sphere, a perpendicular to it, at the point of contact, passes through the center.

COR. 4. If a plane touch a sphere, its line of common section, with the plane of any circle of the sphere passing through the point of contact, is a tangent to that circle.

COR.

COR. 5. A tangent, to any circle of the sphere, is the common tangent of all the circles in whose plane it is.

PROPOSITION V.

If the angular points of any spherical triangle be made the poles of three great circles, another triangle will be formed by their intersections, such, that the sides of the one triangle will be respectively the supplements of the measures of the angles opposite to them in the other.

Fig. 5.

Let the angular points of the triangle ABC be the poles of three great circles, which, by their intersections, form the three lunary surfaces, DQ, FR, and EO; A being the pole of EF, B the pole of DF, and C the pole of ED. Then the triangle DEF, which is common to these three lunary surfaces, will be, in every respect, supplemental to the triangle ABC.

For, let each side of ABC be produced, to meet the sides that contain the angle opposite to it, in the triangle DEF. Then, because BC passes through the poles of ED, DF; ED, DF must also pass through the poles of BC^a: Therefore the points D, Q are the poles of BC. In like manner, R, F are the poles of AB, and E, O, the poles of AC. Hence EL, FK are quadrants^b, and therefore EF the supplement of KL.

But,

^a 7. c. 1.

^b 1. c. 1.

But, since A is the pole of EF, KL is the measure of the angle at A°. Thus, EF is the supplement of the measure of the angle at A. In like manner, FD is the supplement of the measure of the angle at B, and DE the supplement of the measure of the angle at C.

Further, it will appear in the same manner, that BC is the supplement of HM, the measure of the angle at D, that AB is the supplement of NK, the measure of the angle at F, and that AC is the supplement of GL, the measure of the angle at E.

COR. 1. In each of the three triangles, which, with the supplemental triangle, make up the three lunar surfaces, two of the sides are the measures of two angles in the original triangle; and conversely. Also, the third side, and its opposite angle, are the supplements of the third angle, and its opposite side. They are named semi-supplemental triangles.

COR. 2. Let a great circle pass through D, A, the vertices of the triangles ABC, DEF, it will cut the bases BC, EF at right angles^d, because it passes thro' their poles, and the segments BP, PC, of the base, in the one triangle, will be the complements of the measures of the vertical angles IDF, IDE, in the other, taken alternately.

COR. 3. If the original triangle be right angled, the supplemental and semi-supplemental, are quadrantal

tal triangles ; or, if the former be quadrantal, the latter are right angled.

PROPOSITION VI.

Of all the arches of great circles that can be drawn, from a point in the surface of a segment of the sphere that is not the pole of the base, to meet its circumference, that which passes through the pole is the greatest, the other, in the same plane with it, is the least, and that which makes a less angle with the former is the greater of two others.

Fig. 6.

Let AGC be a segment of the sphere, cut off by the plane of the circle ABCD ; also, let G be the pole of the base, and E any other point in the surface of the segment. Of all the arches of great circles which fall from E, upon the circumference of the base, EGA, which passes through the pole, is the greatest, EC, in the same plane with EGA, is the least, and EK, nearer to EGA, is greater than EB, more remote.

Because the plane AGEK passes through the center of the sphere, and one of the poles of the circle ABC, it is perpendicular to the plane of that circle, and passes through its center F. Thus, AFC, their
line

* 5. c. 1.

line of common section, is a diameter of ABC. Let EH be perpendicular to AC, and EA, EK, EB, EC, HK, HB be drawn. Then, of all the straight lines drawn from H to the circumference ABCD, HFA is the greatest, HC the least, and HK greater than HB. Hence, in the right angled triangles EHA, EHK, EHB, EHC, which have the side EH common, EA is the greatest hypotenuse, EC the least, and EK greater than EB. Consequently the arch EA is the greatest, EC the least, and EK greater than EB.

COR. 1. Of all the arches of great circles that fall on the circumference of another great circle, the perpendiculars are the greatest and least, and only two equal arches can fall from the same point, one on each side of the plane of the perpendiculars, and they meet it at equal distances from the perpendicular.

COR. 2. In every isosceles triangle, of which the equal sides are not quadrants, the perpendicular from the vertex bisects the base, and, conversely, the arch from the vertex which bisects the base is perpendicular to it.

PROPOSITION VII.

Any two sides of a spherical triangle are together greater than the third, and all the three sides are together less than a circle.

In

Fig. 7. In the spherical triangle ABG , any two sides are together greater than the third. If the three sides be equal, or if two of them be equal, and each of them greater than the third, the proposition is evident. In every other case, let AC be the greatest, and let A be the pole of the circle DCE , meeting AB produced in D, E . Then BC is greater than BE , and consequently the sum of AB, BC , greater than AE or AC .

Further, all the three sides AB, BC, CA , are together less than a great circle.

For, if AB, BC be produced to meet in F ; BAF, BCF are semi-circles; and CA being less than the sum of AF, FC , the sum of AB, BC, CA will also be less than the sum of BAF, BCF , that is, less than a whole circle.

PROPOSITION VIII.

If two sides of a spherical triangle be equal, the angles opposite to them are also equal; and conversely.

Fig. 8. In the triangle ABC , if the sides AB, AC be equal, the angles ABC, ACB will also be equal. If AB, AC be quadrants, ABC, ACB are right angles. If not, let the tangent to the side AB at B , meet EA , the line of common section of the planes AB, AC , in F ;

F; and let the tangents to the base BC, at its extremities, meet each other in G; also, let FC, FG, EC, and EB, be joined. Then the triangles FEB, FEC have FE common, EB = EC, and the angle AEB = AEC; therefore FB = FC, and the angle FCE = FBE, a right angle. Hence FC is a tangent, and the triangles FGB, GCF mutually equilateral. Therefore the angle FBG = FCG, and consequently ABC = ACB.

Again, if the angles ABC, ACB be equal, the sides AB, AC are equal.

For, in the semi-supplemental triangle EQF, the sides FQ, QE being the measures of the angles B, C, will be equal; therefore the angles at F, E, and consequently their measures AB, AC, are also equal.

COR. 1. Every equilateral spherical triangle is also equiangular; and conversely.

COR. 2. In any triangle, the greater angle is subtended by the greater side; and conversely. If the angle ACB be greater than ABC, let BCD = ABC, then BD = DC, and AB = AD + DC, which is greater than AC.

COR. 3. In every isosceles triangle, of which the equal sides are not quadrants, the perpendicular from the vertex bisects the vertical angle, also, the arch which bisects the vertical angle bisects the base at right angles.

For, since the angles B, C are equal, the supplemental triangle DEF will be isosceles; and therefore

Z z

the

the segments of the base EI, IF, made by the perpendicular, are equal ^a, and their complements ^b IL, IK equal, which are the measures of the vertical angles PAC, PAB. The converse appears in the same manner.

^a 2. c. 1.
^b 2. c. 5.

PROPOSITION IX.

All the angles of any spherical triangle are together greater than two, and less than six right angles.

Fig. 5.

In the triangle ABC, the three angles are together less than six right angles, because, when added to the three exterior angles, they only make six. And they are greater than two right angles, because their measures GH, KL, MN, added to DE, EF, FD, the sides of the supplemental triangle, are equal to three semicircles; and DE, EF, FD, being less than two semicircles, GH, KL, MN, must be greater than one.

COR. I. The sum of all the angles of a spherical triangle, and double the supplement of one of them, is less than six right angles.

For GH, KL, MN, added to DE, EF, FD, being equal to three semicircles, or six quadrants, and EF, FD, greater than DE; GH, KL, MN, added to twice DE, will therefore be less than six quadrants.

COR.

COR. 2. Hence, if one angle of a spherical triangle be right, the sum of the other two is greater than one, but less than three right angles.

COR. 3. If one angle of a spherical triangle be acute, all the three angles are together less than four right angles.

COR. 4. If one angle of a spherical triangle be half a right angle, or less, all the three angles are together less than three right angles.

COR. 5. The exterior angle of any spherical triangle is less than the sum of the two interior opposite angles.

PROPOSITION X.

Any two angles of a spherical triangle are together greater, equal, or less, than two right angles, according as the sum of the opposite sides is greater, equal, or less, than a semicircle; and conversely.

Let the sides AB, AC, of the spherical triangle ABC, be produced to meet in D. Then it is manifest that, according as the sum of AB, BC, is greater, equal, or less than the semicircle ABD; the side BC will be greater, equal, or less, than BD; the angle D (or A) will be greater, equal, or less than BCD; and the sum of the angles BAC, BCA greater, equal, or less, than the sum of BCA, BCD, which is two right angles.

COR.

COR. 1. According as half the sum of any two sides of a spherical triangle is greater, equal, or less, than a quadrant, half the sum of the two opposite angles will be greater, equal, or less, than a right angle; and conversely.

COR. 2. According as one of the equal sides of an isosceles triangle is greater, equal, or less than a quadrant, its opposite angle will be greater, equal, or less than a right angle.

PROPOSITION XL

In a right angled spherical triangle, according as either of the sides about the right angle is greater, equal, or less than a quadrant, its opposite angle is greater, equal, or less than a right angle; and conversely.

Fig. 11.

Let ABC be a triangle right angled at B, and let the sides AB, BC be produced to meet in D. Then, because they pass through each other's poles, E, the middle of BAD, will be the pole BCD. Let a great circle pass through the points CE. The arch EC is a quadrant, and the angle ECB a right angle. Now, it is plain that, according as AB is greater, equal, or less, than the quadrant EB, the opposite angle ACB will be greater, equal, or less, than the right angle ECB; and conversely.

COR.

COR. 1. If the two sides be both greater, or both less than quadrants, the hypotenuse will be less than a quadrant; but, if the one be greater, and the other less, the hypotenuse will be greater than a quadrant; and conversely.

For, of the triangles ABC , ADC , right angled at B , D , in which the sides AB , BC , are less, and consequently AD , DC greater than quadrants, the hypotenuse AC is less than a quadrant, because it is nearer to BC than the quadrant CE . But, in the triangle aBC , of which the side aB is greater, and BC less than a quadrant, the hypotenuse aC is greater than a quadrant, because it is further from CB , than CE is.

COR. 2. In every spherical triangle, of which the two sides are not both quadrants, if the perpendicular from the vertex fall within, the angles at the base will be both acute or both obtuse; but, if it fall without, the one will be obtuse, and the other acute; and conversely.

PROPOSITION XII.

Any two circles, of the sphere, passing through the poles of two great circles, intercept equal arches upon them.

Let

Fig. 12.

Let AFB , CFD be the two great circles intersecting one another in F , and let F be the pole of the great circle $ACBD$, cutting them in the diameters AEB , CED . The circle $ACBD$ passes through their poles; let the diameter MN be perpendicular to AB , and PQ to CD ; then M , N are the poles of AFB , and P , Q , the poles of CFD . Let the small circle PGN pass through the poles P , N , and cut the circle $ACBD$ in the line of common section $PKLN$; the arches BH , DG , of the circles AFB , CFD , intercepted by the circle passing through P , N , are equal.

For, let EH , NK , EG , GL , PG , NH , be drawn.

The triangles PEL , NEK , are equal in every respect; hence $PL = NK$, and $EL = EK$.

The triangles PEG , NEH , are equal; therefore $PG = NH$; hence $PH = NG$, and $PNH = NPG$.

The triangles NKH , PLG are equal; therefore $HK = LG$.

The triangles EKH , ELG are equal; therefore the angle $HEK = GEL$, and $HB = GD$. Hence the arches intercepted between any two circles passing through P , N are equal.

APPENDIX.

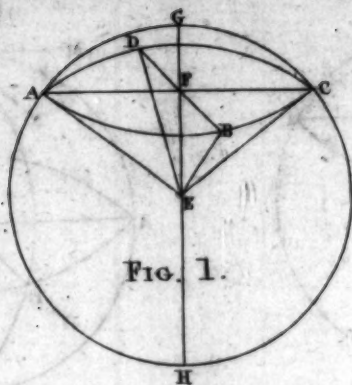


FIG. 1.

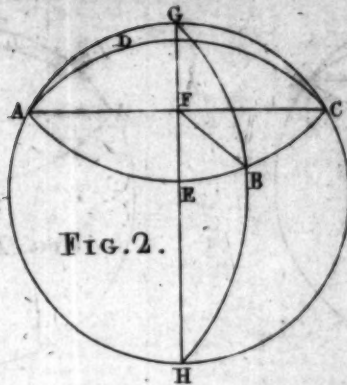


FIG. 2.

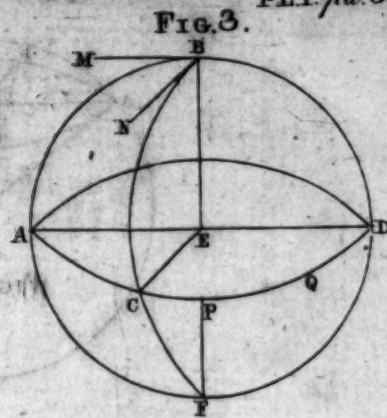


FIG. 3.

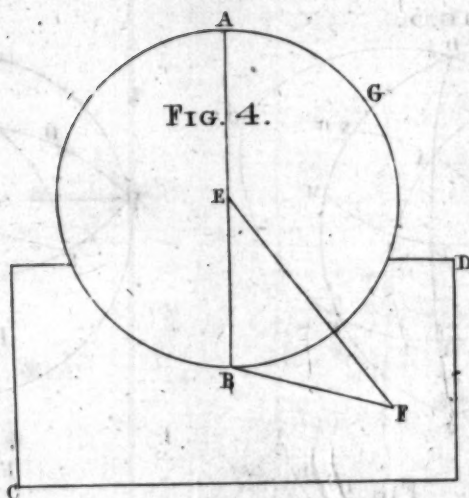


FIG. 4.

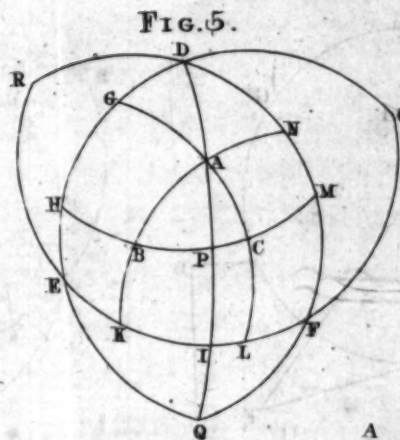


FIG. 5.

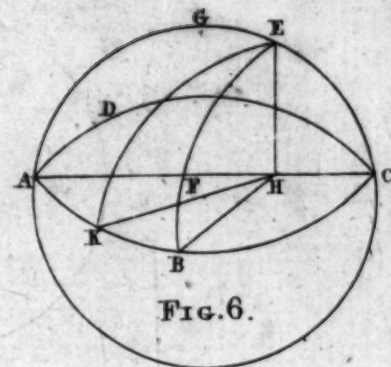


FIG. 6.

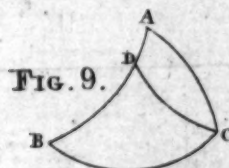


FIG. 9.

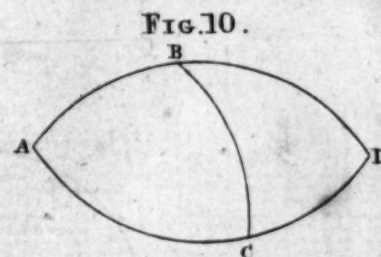


FIG. 10.

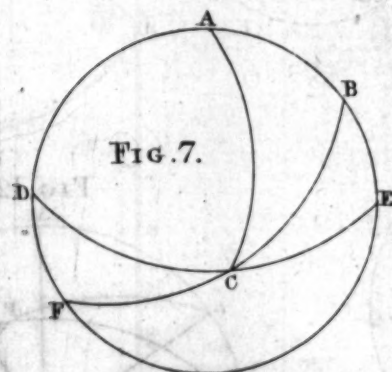


FIG. 7.

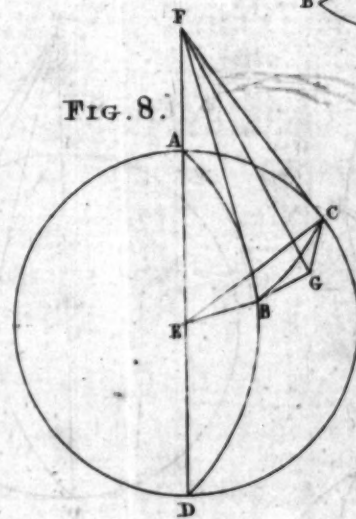


FIG. 8.

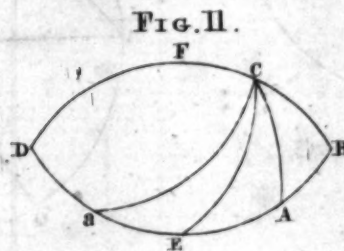


FIG. 11.

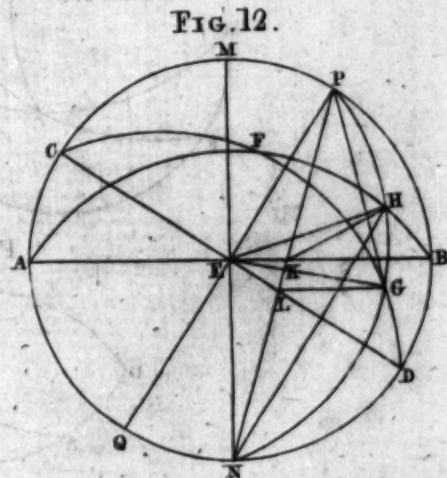


FIG. 12.

A P P E N D I X

Prop. 1. Two spherical triangles, which have two sides of the one equal to two sides of the other, each to each, and the included angles also equal, are equal in every respect.

Prop. 2. Two spherical triangles, which have two angles of the one equal to two angles of the other, each to each, and the included sides also equal, are equal in every respect.

Prop. 3. Two spherical triangles, which have the three sides of the one equal to the three sides of the other, each to each, are equal in every respect.

Prop. 4. Two spherical triangles, which have the three angles of the one equal to the three angles of the other, each to each, are equal in every respect.

Prop. 5. Two spherical triangles, which have two sides of the one equal to two sides of the other, each to each, and an angle equal to an angle opposite to equal sides, are equal in every respect, when the other two angles opposite to equal sides are together greater or less than two right angles.

Prop. 6. Two spherical triangles, which have two angles of the one equal to two angles of the other, each to each, and a side equal to a side opposite to equal angles, are equal in every respect, when the

A P P E N D I X.

PROP. 1. Two spherical triangles, which have two sides of the one equal to two sides of the other, each to each, and the included angles also equal, are equal in every respect.

PROP. 2. Two spherical triangles, which have two angles of the one equal to two angles of the other, each to each, and the included sides also equal, are equal in every respect.

PROP. 3. Two spherical triangles, which have the three sides of the one equal to the three sides of the other, each to each, are equal in every respect.

PROP. 4. Two spherical triangles, which have the three angles of the one equal to the three angles of the other, each to each, are equal in every respect.

PROP. 5. Two spherical triangles, which have two sides of the one equal to two sides of the other, each to each, and an angle equal to an angle opposite to equal sides, are equal in every respect, when the other two angles opposite to equal sides, are together greater or less than two right angles.

PROP. 6. Two spherical triangles, which have two angles of the one equal to two angles of the other, each to each, and a side equal to a side opposite to equal angles, are equal in every respect, when
the

the other two sides, opposite to equal angles, are together greater or less than a semicircle.

PROP. 7. In every spherical triangle, if one of the angles be acute, one of the sides is less than a quadrant; or, if one of the sides be greater than a quadrant, one of the angles is obtuse.

PROP. 8. In every spherical triangle, if the three angles be acute, each side is less than a quadrant; or, if each side be greater than a quadrant, the three angles are obtuse.

SPHERICS.

SPHERICAL TRIGONOMETRY.

BOOK II.

SPHERICAL TRIGONOMETRY.

PROPOSITION I.

IN any right angled spherical triangle, as radius is to the sine of the hypotenuse, so is the sine of one of the oblique angles to the sine of its opposite side.

Let ABC be a spherical triangle, having a right angle at B, and let AD, BD, CD, be drawn to the center of the sphere. From C, in the plane DCA, let CE be drawn perpendicular to DA, and from E, in the plane DBA, EF perpendicular to the same line, and let CF be joined. Then, because DA is perpendicular

Fig. 13.

A a a

dicular

dicular to the two lines CE, EF, it is perpendicular to the plane CEF, and consequently the plane CEF perpendicular to the plane DBA. But the plane DCB is also perpendicular to DBA; therefore their line of common section CF is perpendicular to the same. Hence CFD, CFE, are right angles. Now, in the right angled plane triangle CFE, $R : CE :: S, E : CF$. But the angle CEF, being the inclination of the planes DCA, DBA, is the same with the spherical angle CAB, CE is the sine of AC, and CF the sine of BC; therefore $R : S, AC :: S, A : S, BC$.

COR. 1. As radius is to the cosine of one of the sides, so is the cosine of the other to the cosine of the hypothenuse.

Fig. 14.

For, let the great circle, of which A is the pole, meet the three sides in D, E, F. Then F is the pole of AD; and, applying this proposition to the complementary triangle FCE, $R : S, FC :: S, F : S, CE$, that is, $R : \text{Cos. } BC :: \text{Cos. } AB : \text{Cos. } AC$.

COR. 2. As radius is to the cosine of one of the sides, so is the sine of its adjacent angle to the cosine of the other angle.

PROPO.

PROPOSITION II.

In any right angled spherical triangle, as radius is to the sine of one of the sides, so is the tangent of its adjacent angle to the tangent of the other side.

From B, let BE be drawn perpendicular to DA, Fig. 15.
and from E, EF also perpendicular to DA, in the plane DCA, to meet DC in F, and let FB be joined. It may be shewn, as in the preceding proposition, that FB is perpendicular to the plane DBA. Hence FB is the tangent of BC, and FBE a right angled triangle; therefore $R : EB :: T, E : FB$; that is, $R : S, AB :: T, A : T, BC$.

COR. 1. As radius is to the cosine of the hypothenuse, so is the tangent of one of the angles to the cotangent of the other.

For, in the complementary triangle FCE, $R : S, CE :: T, C : T, FE$; that is, $R : \text{Cos. } AC :: T, C : \text{Cot. } A$, or, $R : \text{Cos. } AC :: T, A : \text{Cot. } C$.

COR. 2. As radius is to the cosine of one of the angles, so is the tangent of the hypothenuse to the tangent of the side adjacent to that angle.

For $R : S, FE :: T, F : T, CE$; that is, $R : \text{Cos. } A :: \text{Cot. } AB : \text{Cot. } AC$, or $R : \text{Cos. } A :: T, AC : T, AB$.

S C H O.

S C H O L I U M.

Let the hypotenuse, the two angles, and the complements of the two sides of any right angled spherical triangle, be named the five circular parts of the triangle. Any one of these being considered as the *middle part*, let the two which are next to it be called the *adjacent parts*, and the remaining two the *opposite parts*. Then the two preceding theorems, with their corollaries, may be all expressed in one proposition, adapted to practice, as follows.

P R O P O S I T I O N III.

In any right angled spherical triangle, the rectangle under radius and cosine of the middle part, the rectangle under the cotangents of the adjacent parts, and the rectangle under the sines of the opposite parts are all equal.

$$R \times \text{Cos. } M = \text{Cot. } A \times \text{Cot. } a.$$

$$R \times \text{Cos. } M = S, O \times S, o.$$

Fig. 16.

CASE I. Let the hypotenuse AC be the middle part.

Then

Then $^a R : \text{Cos. AC} :: T, C : \text{Cot. A.}$ $^a 1. c. 2.$
 Therefore $(R : T, C ::) \text{Cot. C} : R :: \text{Cos. AC} : \text{Cot. A.}$
 And $^b R : \text{Cos. AB} :: \text{Cos. BC} : \text{Cos. AC.}$ $^b 1. c. 1.$

CASE II. Let the angle A be the middle part.

Then $^c R : \text{Cos. A} :: T, AC : T, AB.$ $^c 2. c. 2.$
 Therefore $(R : T, AC ::) \text{Cot. AC} : R :: \text{Cos. A} : T, AB.$
 And $^d R : \text{Cos. BC} :: S, C : \text{Cos. A.}$ $^d 2. c. 1.$

CASE III. Let the complement of the side AB be the middle part.

Then $^e R : S, AB :: T, A : T, BC.$ $^e 2.$
 Therefore $(R : T, A ::) \text{Cot. A} : R :: S, AB : T, BC.$
 And $^f R : S, AC :: S, C : S, AB.$ $^f 1.$

S C H O L I U M.

With respect to right angled triangles, the general problem in spherical trigonometry is, When any two parts besides the right angle are given, to find the three remaining parts. Of which there are the following cases.

CASE 1. When the hypotenuse and one of the angles are given.

CASE 2. When one of the sides and one of its adjacent angles are given.

CASE

CASE 3. When one of the sides, and its opposite angle, are given.

CASE 4. When the hypotenuse and one of the sides are given.

CASE 5. When the two sides are given.

CASE 6. When the two angles are given.

In applying the above proposition to the solution of these cases, let each of the given parts be made the middle part, by which two of the required parts will be found; and, to find the remaining one, let the same be assumed as the middle part.

By Prop. XI. and COR. 1. B. I. each of the unknown parts is in every case, except the 3d, limited to one value.

PROPOSITION IV.

In any spherical triangle, the sines of the sides are directly proportional to the sines of the opposite angles.

This proposition has been demonstrated in the case of right angled triangles.

Fig. 17. Let ABC be any oblique angled triangle, divided into two right-angled triangles ABD, CBD, by the perpendicular BD, falling from the vertex upon the base AC.

In

In the former, the complement of BD being the middle part, $R \times S, BD = S, AB \times S, A^2$. In the latter, the complement of BD being the middle part, $R \times S, BD = S, BC \times S, C$. Hence $S, AB \times S, A = S, BC \times S, C$, and $S, AB : S, BC :: S, C : S, A$.

COR. 1. The cosines of the two sides are to one another directly as the cosines of the segments of the base.

For this may be proved in like manner, by making AB, BC the middle parts.

COR. 2. The tangents of the two sides are to one another inversely as the cosines of the vertical angles. This will follow from making the angles ABD, CBD the middle parts.

L E M M A I.

The sum of the tangents of two arches is to their difference, as the rectangle under the sine and cosine of half their sum to the rectangle under the sine and cosine of half their difference.

Let AB and AC (= AE) be the two arches, AG, AF their tangents, D the center, DK, DI perpendicular to BC, BE. Then the triangle GDF : BDC ($:: GD \times DF : BD \times DC :: GD \times DH : BD \times DE ::$) GDH : BDE.

And

And alternately $GDF : GDH :: BDC : BDE$.

Therefore $FG : GH :: BK \times KD : BI \times ID$.

That is, $T, A + T, a : T, A - T, a :: S, \frac{A + a}{2} \times \text{Cos. } \frac{A + a}{2} : S, \frac{A - a}{2} \times \text{Cos. } \frac{A - a}{2}$.

L E M M A II.

The sum of the sines of two arches is to their difference as the rectangle under the sine of half the sum, and cosine of half the difference of these arches, is to the rectangle under the sine of half the difference, and cosine of half the sum.

Fig. 19.

Let AB and $AD (= AK)$ be the two arches, C the center ; KO, DL , parallel to CA ; CR, CI, BG , perpendicular to BK, BD, DL ; and AH, AM parallel to BD, BK . The right angled triangles BGD, CHA , also BOK, CMA , by reason of parallel lines, are equiangular.

Hence $BG : BD :: CH : CA$, and $BG \times CA = BD \times CH$.

Also, $BO : BK :: CM : CA$, and $BO \times CA = BK \times CM$.

Consequently $BG : BO :: BD \times CH : BK \times CM :: BI \times CH : BR \times CM$.

That

That is, $S, A + S, a : S, A - S, a :: S, \frac{A + a}{2} \times \text{Cos.}$

$$\frac{A - a}{2} : S, \frac{A - a}{2} \times \text{Cos.} \frac{A + a}{2}.$$

L E M M A III.

The sum of the fines of two arches is to their difference as the tangent of half the sum of these arches is to the tangent of half their difference.

Let AB, AD, be the two arches, C the center, CF, Fig. 20.
DO, BQ, perpendicular to CA, also DM, BN perpendicular to CF, BE = ED, and FEK the tangent at E, which will be parallel to BD, because they are both perpendicular to the radius CE. Now, by similar triangles,

$$DO : BQ :: DL : BL :: GK : HK.$$

Therefore $DO + BQ : DO - BQ :: 2EK : 2EH$
 $:: EK : EH.$

That is, $S, A + S, a : S, A - S, a :: T, \frac{A + a}{2}$
 $: T, \frac{A - a}{2}.$

Bbb

LEMMA

L E M M A IV.

The sum of the cosines of two arches is to their difference as the co-tangent of half the sum of these arches to the tangent of half their difference.

For, in the same figure, $BN : DM :: BI : DI :: HF : FG$.

Therefore $BN + DM : BN - DM :: 2EF : 2EG :: EF : EG$.

That is, $\text{Cos. } A + \text{Cos. } a : \text{Cos. } A - \text{Cos. } a ::$

$\text{Cot. } \frac{A+a}{2} : \text{T. } \frac{A-a}{2}$.

P R O P O S I T I O N V.

In any spherical triangle, the tangent of half the sum of the segments of the base is to the tangent of half the sum of the two sides, as the tangent of half their difference to the tangent of half the difference of the segments of the base.

Fig. 17.
I. C. 4.

For, $\text{Cos. } AB : \text{Cos. } BC :: \text{Cos. } AD : \text{Cos. } DC$.

Therefore $\text{Cos. } AB + \text{Cos. } BC : \text{Cos. } AB - \text{Cos. } BC :: \text{Cos. } AD + \text{Cos. } DC : \text{Cos. } AD - \text{Cos. } DC$.

And,

And^b Cot. $\frac{AB + BC}{2} : T, \frac{AB - BC}{2} ::$ Cot. $\frac{AD + DC}{2}$ ^b Lem. 4.

$: T, \frac{AD - DC}{2}.$

Or, Cot. $\frac{AB + BC}{2} : Cot. \frac{AD + DC}{2} :: T, \frac{AB - BC}{2}$

$: T, \frac{AD - DC}{2}.$

Hence, $T, \frac{AD + DC}{2} : T, \frac{AB + BC}{2} :: T, \frac{AB - BC}{2}$

$: T, \frac{AD - DC}{2}.$

COR. 1. The cotangent of half the sum of the vertical angles, and the tangent of half their difference, or the cotangent of half their difference, and the tangent of half their sum, according as the perpendicular falls within or without, are reciprocally proportional to the tangents of half the sum, and half the difference of the angles at the base.

For, let the vertical angles ABD, CBD, or their measures, be denoted by x and y ; then, applying this proposition to the semi-supplemental triangle EQF,

we have $T, \frac{EI + IF}{2} : T, \frac{EQ + QF}{2} :: T, \frac{EQ - QF}{2}$ Fig. 21,

$: T, \frac{EI - IF}{2}.$ But, when the perpendicular falls

within, $EI - IF = x - y$, and $\frac{EI + IF}{2}$ is the com-

plement of $\frac{x + y}{2}$. And, when it falls without, EI

$- IF = x + y$, and $\frac{EI + IF}{2}$ is the complement

of

of $\frac{x-y}{2}$, or, $EI + IF = x + y$, and $\frac{E-IF}{2}$ is the complement of $\frac{x-y}{2}$.

Therefore,

$$1. \text{Cot. } \frac{x+y}{2} : T, \frac{A+C}{2} :: T, \frac{A-C}{2} : T, \frac{x-y}{2}$$

$$2. \text{Cot. } \frac{x-y}{2} : T, \frac{A+C}{2} :: T, \frac{A-C}{2} : T, \frac{x+y}{2}$$

COR. 2. Hence,

$$1. \text{Cot. } \frac{x+y}{2} : T, \frac{A+C}{2} :: \text{Cot. } \frac{x+y}{2} \times T, \frac{A-C}{2} : T, \frac{A+C}{2} \times T, \frac{x-y}{2}$$

$$2. \text{Cot. } \frac{x-y}{2} : T, \frac{A+C}{2} :: \text{Cot. } \frac{x-y}{2} \times T, \frac{A-C}{2} : T, \frac{A+C}{2} \times T, \frac{x+y}{2}$$

COR. 3. Also,

$$1. \text{Cot. } \frac{x+y}{2} : T, \frac{A-C}{2} :: \text{Cot. } \frac{x+y}{2} \times T, \frac{A+C}{2} : T, \frac{A-C}{2} \times T, \frac{x-y}{2}$$

$$2. \text{Cot. } \frac{x-y}{2} : T, \frac{A-C}{2} :: \text{Cot. } \frac{x-y}{2} \times T, \frac{A+C}{2} : T, \frac{A-C}{2} \times T, \frac{x+y}{2}$$

PROPO.

VI.

In any spherical triangle, the sine of half the sum of the two sides, is to the sine of half their difference, as the co-tangent of half the vertical angle to the tangent of half the difference of the angles at the base.

Fig. 17.

2. C. 4.

$$\cos. x + \cos. y : \cos. x - \cos. y.$$

Hence $\therefore S_1 \frac{AB+BC}{2} \times \cos. \frac{AB+BC}{2} : S_2 \therefore \text{Lem. 1}$

$$\frac{AB - BC}{2} \quad \text{Cos. } \frac{AB - BC}{2} :: \text{Cot. } \frac{x+y}{2} : T, \frac{x-y}{2}$$

or :: Cot. $\frac{x-y}{2}$: T, $\frac{x+y}{2}$.

C 4.

Therefore $S, AB + S, BC : S, AB - S, BC ::$

$$S, A + S, C : S, A - S, C.$$

Hence $\angle S, \frac{AB + BC}{2} \times \cos. \frac{AB - BC}{2} : \cos. \angle Lem. 3,$

$$\frac{AB + BC}{2} \times S, \frac{AB - BC}{2} :: T, \frac{A + C}{2} : T, \frac{A - C}{2}.$$

Consequently $S, \frac{AB + BC}{2} : S, \frac{AB - BC}{2} :: \text{Cot.}$

3. c. 5.

PROPO.

PROPOSITION VII.

In any spherical triangle, the cosine of half the sum of the two sides, is to the cosine of half their difference, as the co-tangent of half the vertical angle to the tangent of half the sum of the angles at the base.

Fig. 17. For, in the demonstration of the preceding proposition, we have these analogies.

$$S, \frac{AB + BC}{2} \times \text{Cos. } \frac{AB + BC}{2} : S, \frac{AB - BC}{2} \\ \times \text{Cos. } \frac{AB - BC}{2} :: \text{Cot. } \frac{x + y}{2} : \text{Cot. } \frac{x - y}{2}.$$

$$S, \frac{AB - BC}{2} \times \text{Cos. } \frac{AB + BC}{2} : S, \frac{AB + BC}{2} \\ \times \text{Cos. } \frac{AB - BC}{2} :: T, \frac{A - C}{2} : T, \frac{A + C}{2}.$$

$$\text{Whence, Cos. } \frac{AB + BC}{2} : \text{Cos. } \frac{AB - BC}{2} :: \text{Cot.}$$

$$\frac{1}{2} B : T, \frac{A + C}{2}.$$

PROPO.

PROPOSITION VIII.

In any spherical triangle, the sine of half the sum of the angles at the base is to the sine of half their difference, as the tangent of half the base to the tangent of half the difference of the two sides.

For, in the triangle EQF, whose sides FQ, QE are the measures of the angles A, C, and whose angles F, E are measured by the sides AB, BC; also, whose base EF is the supplement of the measure of the angle at B, and whose angle at Q is measured by the supplement of the base AC. Fig. 21.

$$S, \frac{EQ + QF}{2} : S, \frac{EQ - QF}{2} :: \text{Cot. } \frac{1}{2} Q : T, \frac{E - F}{2}.$$

That is, $S, \frac{A + C}{2} : S, \frac{A - C}{2} :: T, \frac{1}{2} AC : T, \frac{AB - BC}{2}.$

PROPOSITION IX.

In any spherical triangle, the cosine of half the sum of the angles at the base, is to the cosine of half their difference, as the tangent of half the base to the tangent of half the sum of the two sides.

For,

For, in the semi-supplemental triangle EQF,

$$\text{Cos. } \frac{EQ + QF}{2} : \text{Cos. } \frac{EQ - QF}{2} :: \text{Cot. } \frac{1}{2} Q \\ : T, \frac{E + F}{2}.$$

$$\text{That is, Cos. } \frac{A + C}{2} : \text{Cos. } \frac{A - C}{2} :: T, \frac{1}{2} AC : T, \\ \frac{AB + BC}{2}.$$

S C H O L I U M.

Let one of the six parts of any spherical triangle be neglected; let the one opposite to it, or its supplement, if an angle, be called the *middle part*, the two next to it the *adjacent parts*, and the remaining two the *opposite parts*. Then the four preceding propositions may be expressed in more general terms, as follows:

PROPOSITION. X.

In any spherical triangle, the sine or cosine of half the sum of the adjacent parts, is to the sine or cosine of half their difference, as the tangent of half the middle part to the tangent of half the difference, or half the sum of the opposite parts.

FIG. 13.

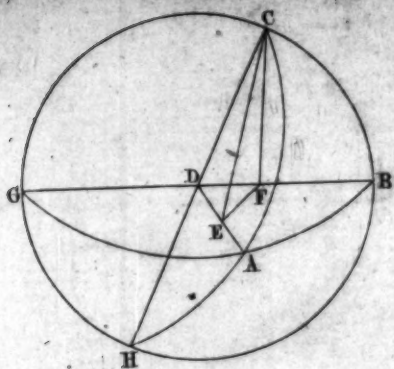


FIG. 14.

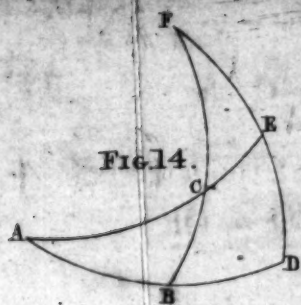


FIG. 15.

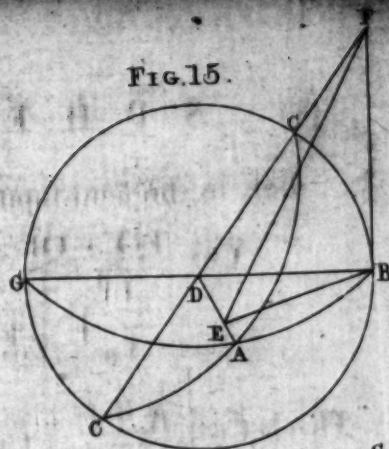


FIG. 16.

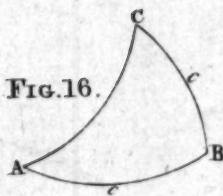


FIG. 17.

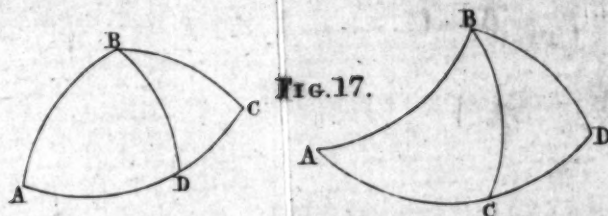


FIG. 19.

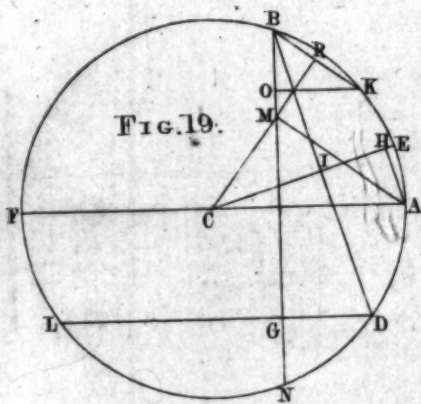


FIG. 20.

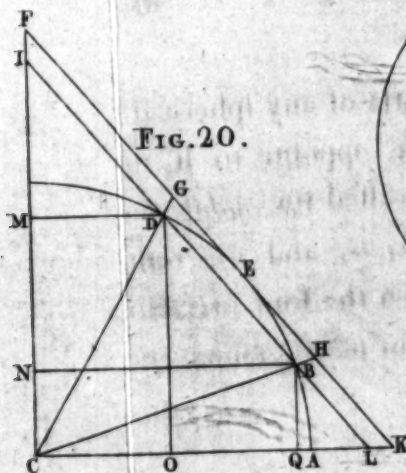


FIG. 18.

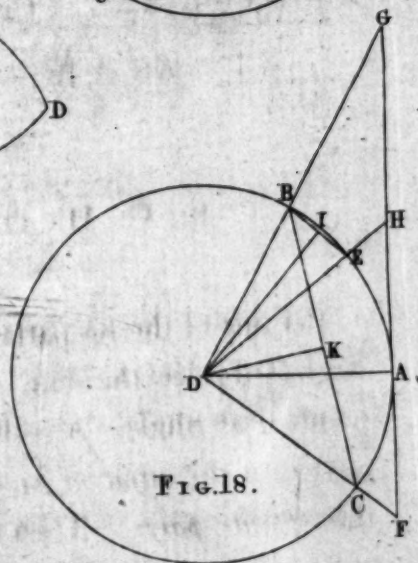


FIG. 21.

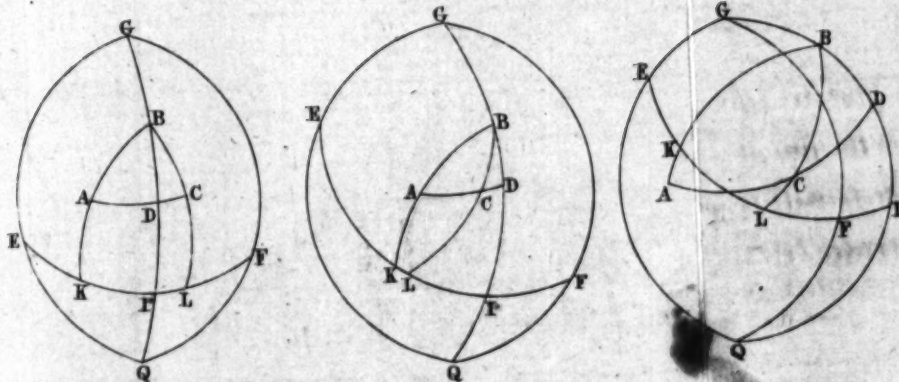
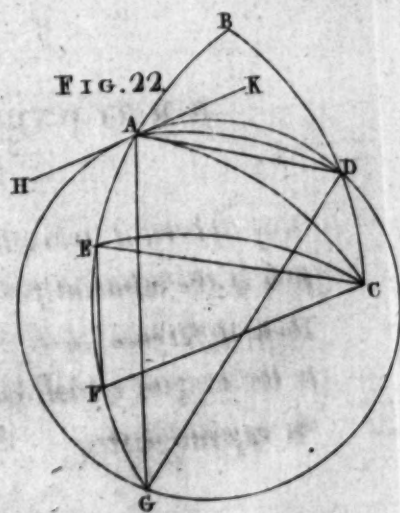


FIG. 22.



THESE ARE THE

REMARKS

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ON

$$S, \frac{A+a}{2} : S, \frac{A-a}{2} :: T, \frac{1}{2} M : T, \frac{O-o}{2}.$$

$$\text{Cos. } \frac{A+a}{2} : \text{Cos. } \frac{A-a}{2} :: T, \frac{1}{2} M : T, \frac{O+o}{2}.$$

PROPOSITION XL

If, from the extremities of the base of any spherical triangle, arches of great circles be described to meet the sides, and to cut off a part on each from the vertex, equal to the other side; the rectangle under the sines of half these arches shall be equal to the rectangle under the sines of half the sum and half the difference of the base, and the difference of the two sides.

Let ABC be a spherical triangle, having two unequal sides AB, BC. From BC, the greater, let BD be cut off equal to BA, and from BA produced BE = BC; and let great circles pass thro' A, D, and E, C.

Fig. 22.

Then, $S, \frac{1}{2} AD \times S, \frac{1}{2} EC = S, \frac{AC + AE}{2} \times S, \frac{AC - AE}{2}.$

For, in BA produced, let AF = AC, and FG = AE; also, let the straight lines AG, GD, DA, EF, FC, CE, be drawn. The straight lines AD and EC are
C c c
parallel;

parallel; for, if a great circle bisect the angle at B, it also bisects the arches AD, EC, and is perpendicular to their planes; therefore the chords AD, EC are perpendicular to the lines of common section, and consequently both are perpendicular to the plane of that great circle. But AG and EF are also parallel. Therefore the plane of the triangle DAG is parallel to the plane of the triangle CEF. Let the plane AFC, which cuts the latter in FC, cut the former in HK. Then HK is parallel to FC. Therefore, since the chords AF, AC are equal, HK is a tangent to the circle that passes through A, F, C, and consequently it is also a tangent to the circle that passes through

5. c. 4. 1. A, D, G. Hence the angle $\text{ADG} = \text{GAH} = \text{EFC}$. But the angle $\text{DAG} = \text{CEF}$. Therefore the triangles AGD, EFC are equiangular, $\text{AD} : \text{AG} :: \text{EF} : \text{EC}$
 $\frac{1}{2} \text{AD} : \frac{1}{2} \text{AG} :: \frac{1}{2} \text{EF} : \frac{1}{2} \text{EC}$, and $\frac{1}{2} \text{AD} \times \frac{1}{2} \text{EC} = \frac{1}{2} \text{AG} \times \frac{1}{2} \text{EF}$.

PROPOSITION III.

In any spherical triangle, the rectangle under the sines of the two sides, is to the square of the radius, as the rectangle under the sines of half the sum, and half the difference of the base, and the difference of the two sides to the square of the sine of half the vertical angle.

Let

Let ABC be the triangle, $BD = BA$, $BE = BC$, and BFG bisecting the vertical angle B; then BFG also bisects AD, EC, and cuts them at right angles. Therefore, in the right angled triangles ABF, EBG.

Fig. 23.

$$S, AB : R :: S, AF : S, \frac{1}{2} B.$$

$$S, BE : R :: S, EG : S, \frac{1}{2} B.$$

$$\text{Hence } S, AB \times S, BC : R^2 :: S, AF \times S, EG : S, \frac{1}{2} B.$$

$$\text{And } S, AB \times S, BC : R^2 :: S, \frac{AC + AE}{2} \times S, \frac{AC - AE}{2} : S, \frac{1}{2} B.$$

S C H O L I U M.

Of the general problem in spherical Trigonometry there are six cases.

CASE 1. When two sides, and an angle opposite to one of them, are given.

CASE 2. When two angles, and a side opposite to one of them, are given.

CASE 3. When two sides, and the included angle are given.

CASE 4. When two angles, and the included side are given.

CASE 5. When the three sides are given.

CASE 6. When the three angles are given.

As

As every oblique angled triangle may be resolved into two right angled triangles, all these cases may be solved, by means of the 3d and 5th propositions only. But a much better solution may be obtained, from the 4th, 10th, and 12th propositions. And the cases may be all reduced to three; for, by using the supplemental or semi-supplemental triangle, in the 2d, 4th, and 6th cases, these may be converted into the 1st, 3d, and 5th, respectively.

By Prop. X. XI. and Cor. each of the unknown parts is limited to one value in all the cases, excepting some of the subcases of the 1st and 2d.

In the third case, the unknown side may be found, without finding the angles, by means of the following Theorem, which is demonstrated in the Appendix to Book IV.

THEOREM. In any spherical triangle, as the square of the radius is to the rectangle under the sines of the two sides, so is the versed sine of the included angle to the excess of the versed sine of the base above the versed sine of the difference of the two sides.

S P H E R I C S.

SPHERICS.

B O O K III.

STEREOGRAPHIC PROJECTION

OF THE SPHERE.

DEFINITIONS.

1. **T**O project an object, is to represent every point of it upon the same plane, as it appears to the eye in a certain position.
2. That plane, upon which the object is projected, is called the plane of projection; and the point where the eye is situated the projecting point.
3. The stereographic projection of the sphere is that in which a great circle is assumed as the plane of projection, and one of its poles as the projecting point.
4. The great circle, upon whose plane the projection is made, is called the primitive.

5. By

5. By the semi-tangent of an arch is meant the tangent of half that arch.

6. By the line of measures of any circle of the sphere is meant that diameter of the primitive, produced indefinitely, which is perpendicular to the line of common section of the circle and the primitive,

Again, let the point C and M be projected into A and O and the whole arch MC will be projected

The projection or representation of any point is where the straight line, drawn from it to the projecting point, intersects the plane of projection.

PROPOSITION I.

Every great circle which passes through the projecting point is projected into a straight line, passing through the center of the primitive; and every arch of it, reckoned from the other pole of the primitive, is projected into its semi-tangent.

Fig. 24.

Let $ABCD$ be a great circle, passing through A, C , the poles of the primitive, and intersecting it in the line of common section BED , E being the center of the sphere. From A , the projecting point, let there be

be drawn straight lines AP, AM, AN, AQ , to any number of points, P, M, N, Q , in the circle $ABCD$. These lines will intersect BED , which is in the same plane with them; let them meet it in the points p, m, n, q ; then p, m, n, q are the projections of P, M, N, Q . And thus the whole circle $ABCD$ is projected into the straight line BED , passing through the center of the primitive.

Again, because the points C and M are projected into E and m , the whole arch MC will be projected into the straight line mE , which, to the radius AE , is the T , $mAE = T \div MC$. Thus, the arch MC is projected into its semi-tangent mE ; PC into its semi-tangent pE , &c. All arches, therefore, on the circle $ABCD$, reckoned from the pole C , are projected into their semi-tangents.

COR. 1. Each of the quadrants contiguous to the projecting point is projected into an indefinite straight line, and each of those that are remote, into a radius of the primitive.

COR. 2. Every small circle which passes through the projecting point is projected into that straight line which is its common section with the primitive.

COR. 3. Every straight line, in the plane of the primitive, and produced indefinitely, is the projection of some circle on the sphere passing through the projecting point.

COR.

COR. 4. *The stereographic projection of any point in the surface of the sphere is distant from the center of the primitive, by the semi-tangent of that point's distance from the pole opposite to the projecting point.*

P R O P O S I T I O N II.

Every circle on the sphere which does not pass through the projecting point is projected into a circle.

Fig. 25.

If the circle be parallel to the primitive, the proposition is evident.

For, a straight line, drawn from the projecting point to any point in the circumference, and made to revolve about the circle, describes the surface of a cone, which is cut by a plane, (viz. the primitive), parallel to the base; and therefore the section (the figure into which the circle is projected) is a circle.

If the circle MN be not parallel to the primitive BD ; let the great circle $ABCD$, passing through the projecting point, cut it at right angles, in the diameter MN , and the primitive in the diameter BD . Through M , in the plane of that great circle, let MF be drawn parallel to BD ; let AM , AN be joined, and meet BD in m , n . Then, because AB , AD are quadrants, and BD , MF parallel, the arch $AM = AN$,

and

and the angle AMF or $AMN = ANM$. Thus, the conic surface, described by the revolution of AM , about the circle MN , is cut by the primitive in a subcontrary position; therefore the section is, in this case, likewise a circle.

COR. 1. The centers, and poles of all circles, parallel to the primitive, have their projections in its center.

COR. 2. The center, and poles of every circle, inclined to the primitive, have their projections in the line of measures.

COR. 3. All projected great circles cut the primitive in two points diametrically opposite; and every circle in the plane of projection, which passes through the extremities of a diameter of the primitive, or through the projections of two points that are diametrically opposite on the sphere, is the projection of some great circle.

COR. 4. A tangent to any circle of the sphere, which does not pass through the projecting point, is projected into a tangent to that circle's projection; also, the circular projections of tangent circles touch one another.

COR. 5. The extremities of the diameter, on the line of measures of any projected circle, are distant from the center of the primitive, by the semi-tangents of the circle on the sphere's least and greatest distances from the pole opposite to the projecting point.

COR. 6. The extremities of the diameter, on the line of measures of any projected great circle, are distant from the center of the primitive, by the tangent and cotangent of half the great circle's inclination to the primitive.

COR. 7. The radius of any projected circle is equal to half the sum, or half the difference, of the Temitangents of the circle's least and greatest distances from the pole opposite to the projecting point, according as the circle does or does not encompass the axis of the primitive.

PROPOSITION III.

An angle, formed by two tangents, at the same point, in the surface of the sphere, is equal to the angle formed by their projections.

Fig. 26.

Let FGI, and GH, be the two tangents, and A the projecting point; let the plane AGF cut the sphere in the circle AGL, and the primitive in the line BML. Also, let MN be the line of common section, of the plane AGH, with the primitive. Then the angle FGH = LMN. If the plane FGH be parallel to the primitive BLD, the proposition is manifest. If not, through any point K, in AG produced, let the plane

plane FKH, parallel to the primitive, be extended to meet FGH in the line FH. Then, because the plane AGF meets two parallel planes, BLD, FKH, the lines of common section LM, FK are parallel; therefore the angle $AML = AKF$. But, since A is the pole of BLD, the chords, and consequently the arches AB, AL are equal; and the arch ABG is the sum of the arches AL, BG. Hence the angle AML is equal to an angle at the circumference, standing upon AG, and therefore equal to AGI, or FGK. Consequently the angle $FGK = FKG$, and the side $FG = FK$. In like manner, $HG = HK$. Hence the triangles GHF, KHF, are equal in every respect, and the angle $FGH = FKH = LMN$.

COR. 1. An angle, contained by any two circles of the sphere, is equal to the angle formed by their projections.

For, the tangents to these circles on the sphere are projected into straight lines, which either coincide with, or are tangents to, their projections on the primitive.

COR. 2. An angle, contained by any two circles of the sphere, is equal to the angle formed by the radii of their projections, at the point of concurrence.

PROPO.

PROPOSITION IV.

The center of a great circle's projection is distant from the center of the primitive by the tangent of the great circle's inclination to the primitive, and its radius is the secant of the same.

Fig. 27.

Let BMDN be the primitive, EA its semi-axis, and A the projecting point; also, let MEN be the diameter of any great circle, oblique to the primitive, and FMGN its projection, the semicircle below the plane of the primitive being projected into the arch MFN, and that above into MGN. Let BED be drawn perpendicular to MN, intersecting the projected circle in F, G; then H, the middle of FG, is its center. Let NH, NF, NG, be drawn, and let NF, NG meet the primitive in L, Q. Then LE, EQ being joined, because GNF is a semicircle, FNQ is a right angle, LBNQ is likewise a semicircle, and LEQ a straight line. Also, let K be the point in the proposed circle, of which G is the projection, and let KE be joined. Then ME, being perpendicular to both AE and EG, is perpendicular to the plane AEG, and consequently to EK. Thus, KE and DE are both at right angles to MN, the line of common section of the great circle and the primitive, and therefore KED is

is its angle of inclination to the primitive. Now, the triangles AEG, NEG, having EG common, AE = EN, and GEA, GEN right angles, are equal in every respect. Thus, the angle EKA = EAK = ENQ = EQN; therefore AEK = NEQ; whence DEQ = DEK = angle of inclination. Again, the angles ENF, EGN are equal; therefore their doubles MEL, EHN are equal; and consequently the complements of these LEB, ENH also equal. Thus, ENH = LEB = DEQ = angle of inclination, and to the radius EN, EH is the tangent, and HN the secant of the angle ENH.

ANOTHER DEMONSTRATION.

Let A be the projecting point, ABCD a great circle passing through it, perpendicular to the proposed great circle, KEL their line of common section, and BED the line of common section of ABCD, and the primitive. Then, because ABCD is perpendicular both to the proposed great circle, and to the primitive, it is perpendicular to their line of common section, and consequently BE, EK are likewise perpendicular to the same. Hence BEK is the angle of inclination of the proposed circle to the primitive. Let AK, AL be drawn, and meet BD in F, G, the straight line FG is the diameter of the projection.

Let

Fig. 28.

Let it be bisected in H, and let A, H be joined. Because FAG is a right angle, $HA = HF$, and the angle $HAF = HEA = FEK + FKE$; from these equals, taking the equal angles EAF, FKE, there remains $HAE = FEK$, the angle of inclination. And, in the right angled triangle AEH, to the radius AE, EH is the tangent, and AH the secant of HAE.

O T H E R W I S E,

Fig. 27.

Let MNG be the projection of a great circle, meeting the primitive in the extremities of the diameter MN, and let the diameter BD, perpendicular to MN, meet the projection in E, G. Let EG be bisected in H, and N, H be joined. Then, because any angle, contained by two circles of the sphere, is equal to the angle formed by the radii of their projections at the point of concurrence, the angle contained by the proposed great circle and the primitive is equal to the angle ENH, of which EH is the tangent, and NH the secant to the radius of the primitive.

Cor. 1. All circles which pass through the points M, N, are the projections of great circles, and have their centers in the line BG. All circles which pass through the points E, G, are the projections of great circles, and have their centers in the line BD. $\angle MEF = \angle MNE = \angle HME = \angle HNE$ (the angle HME = HNE, and the angle HNE = HME) circles, $(\angle MEF +$

circles, and have their centers in the line HI, perpendicular to BG.

COR. 2. If NF, NH be continued to meet the primitive in L, T, then BL is the measure of the great circle's inclination to the primitive, and $MT = 2BL$.

PROPOSITION V.

The center of projection of a small circle, perpendicular to the primitive, is distant from the center of the primitive, the secant of the circle's distance from its nearer pole, and the radius of projection, is the tangent of the same.

Let ABCD be a great circle, passing through the projecting point, and perpendicular to the proposed small circle, MON their line of common section, and BOD the line of common section of ABCD, and the primitive. Then BD is the axis, and O the center of the small circle. Let AM and AN be drawn to meet BD in G, F; EG is the diameter on the line of measures of the small circle's projection. Let it be bisected in H, and EM, MH joined. Then, because $AE : EF :: NO : OF$, $CE : EF :: MO : OF$; therefore the points C, E, M, are in a straight line, and that straight line is perpendicular to AG. Hence $HM = HF$, and the angle HMF ($= HFM = FEM + FME$)

Fig. 29

+ FME) = FEM + FCE ; taking from these the equal angles OMF, FCE, there remains the angle HMO = FEM. Consequently the angle HOM = HME, and HME a right angle. Thus HM, which is equal to the radius of the projection, is the tangent of MD, the distance of the small circle from its nearer pole, and HE is the secant of the same.

Fig. 30.

COR. 1. Let FPGQ be the projection of a small circle, perpendicular to the primitive, and intersecting it in P, Q, and let LEK be the diameter of the primitive, that is, perpendicular to BG, the line of measures : Then L, P, G, or, L, F, Q, are in the same straight line.

For, since $AE : EF :: NO : OF$, $LE : EF :: QO : OF$, &c.

COR. 2. The angle EPH is a right angle, and EP, PH tangents to the two circles.

For, the triangles EMH, EPA are equal in every respect.

COR. 3. Any radius HI is a tangent to the circle which passes through the points B, I, D.

PROPO.

PROPOSITION IV.

The projections of the poles of any circle, inclined to the primitive, are in the line of measures distant from the center of the primitive, the tangent and cotangent of half its inclination.

Because ABCD is perpendicular to the plane of the great circle KL, it passes through its poles, (which are also the poles of all its parallel small circles), let these be p, q , and let Ap, Aq , meet BD in P, Q, their projections. Then, the quadrants pK, CB being equal, and CK common to both, pC will be equal to BK , which measures the inclination of the great circle, (or its parallel small circles), to the primitive. Now, EP, to the radius AE, is the tangent of $\frac{1}{2} pC$, and EQ the tangent of $\frac{1}{2} qC$, or cotangent of $\frac{1}{2} pC$. Fig. 28.

COR. 1. The projection of that pole which is adjacent to the projecting point, is without the primitive, and the projection of the other within.

COR. 2. The distances of either projected pole from the centers of the primitive and projected great circle, are directly proportional to the radii of these circles.

For AP bisects the angle EAH, and consequently AQ bisects the external angle EAR. Hence $EP : PH :: EA : AH$, and $EQ : QH :: EA : AH$.

PROPOSITION VII.

If, from either pole of a projected great circle, two straight lines be drawn to meet the primitive and the projection, they will intercept corresponding arches of these circles.

Fig. 31. From the pole P , of the projected great circle FMG , let there be drawn any two straight lines PL , PQ , meeting the primitive in R , S , and the projection in L , Q ; then shall the arch LQ , be the projection of an arch equal to RS .

For SS , RR are the projections of two circles, each of which passes through A , p , a pole of the primitive, and a pole of the great circle, and which, therefore, intercept equal arches upon them. Now, RS is one of the intercepted arches, and the other is projected into LQ . Hence LQ , RS are corresponding arches.

COR. Hence, if, from the point where the projections of two great circles intersect one another, two straight lines be drawn through their adjacent poles, these will intercept on the primitive an arch, which is the measure of their inclination.

P R O P O.

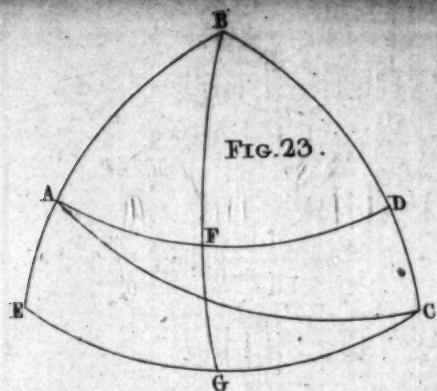


FIG. 23.

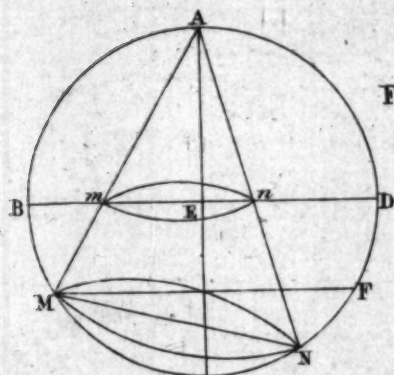


FIG. 25.

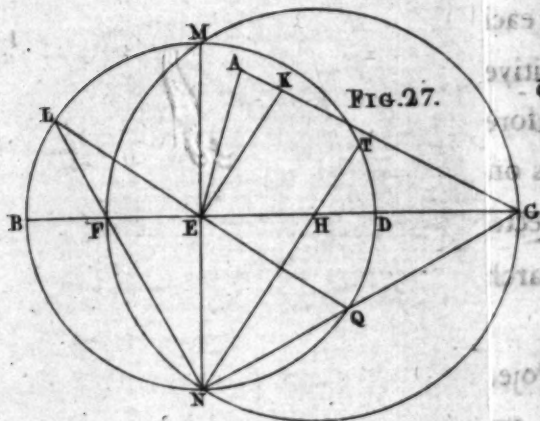


FIG. 27.

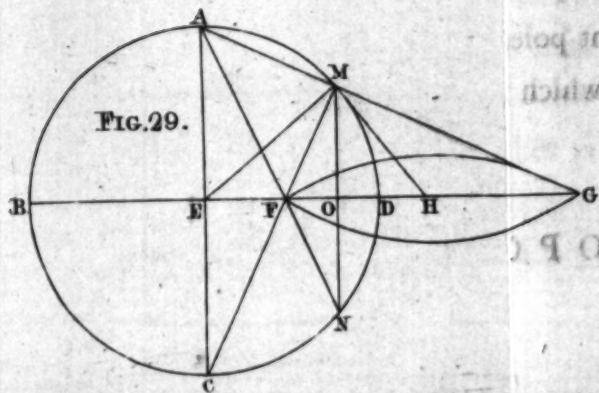


FIG. 29.

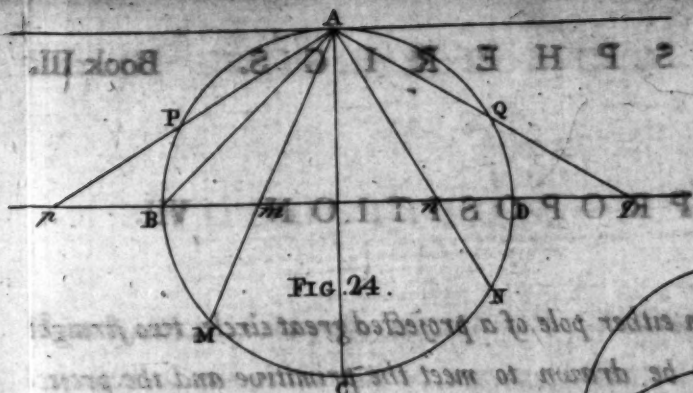


FIG. 24.

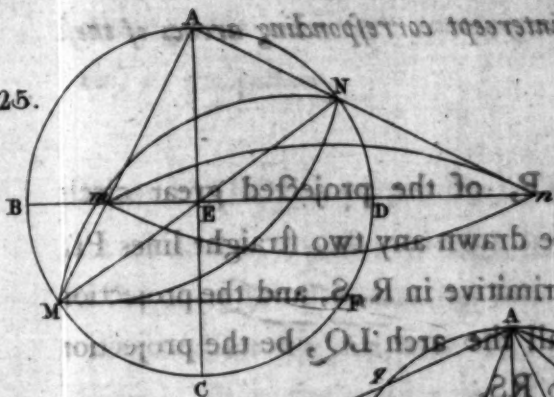


FIG. 26.

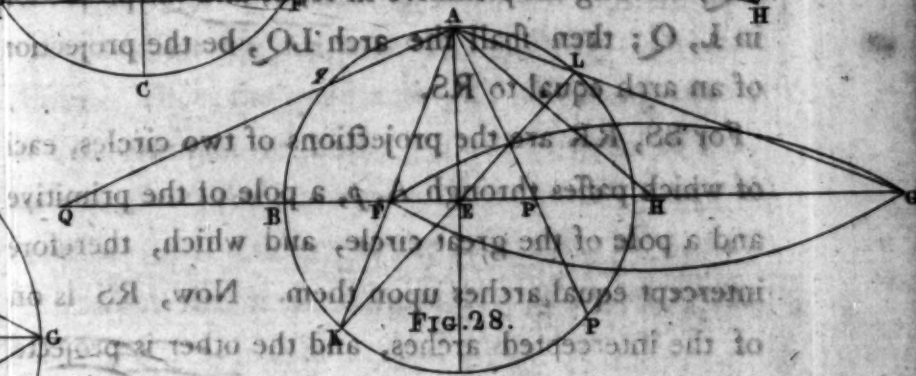


FIG. 28.

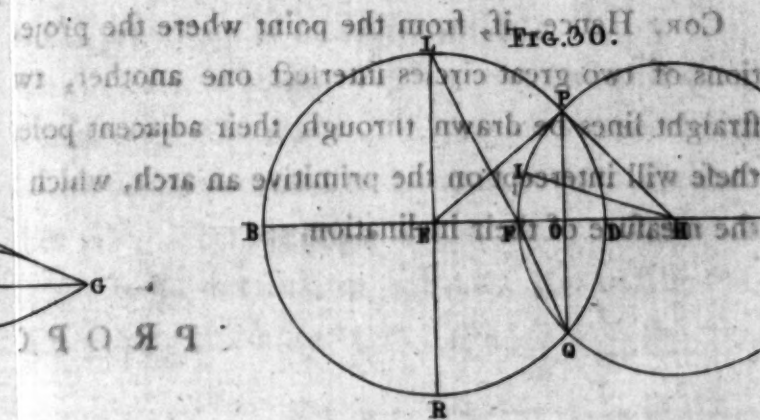


FIG. 30.

PROPOSITION VIII

Through two given points, in the plane of the primitive, to describe the projection of a great circle.

PROPOSITION IX

Through a given point, in the circumference of the primitive, to describe the projection of a great circle that shall form with the primitive a given angle.

PROPOSITION X

Through a given point in the circumference of a projected great circle, to describe the projection of another great circle, that shall make with the former a given angle.

PROPOSITION XI

To describe the projection of a small circle, parallel to the primitive, in distance from the projecting point being given.

PROPO

PROPOSITION VIII.

Through two given points, in the plane of the primitive, to describe the projection of a great circle.

PROPOSITION IX.

Through a given point, in the circumference of the primitive, to describe the projection of a great circle that shall form with the primitive a given angle.

PROPOSITION X.

Through a given point in the circumference of a projected great circle, to describe the projection of another great circle, that shall make with the former a given angle.

PROPOSITION XI.

To describe the projection of a small circle, parallel to the primitive, its distance from the projecting point being given.

PROPO.

PROPOSITION XII.

To describe the projection of a small circle, perpendicular to the primitive, its pole, and distance from its pole, being given.

PROPOSITION XIII.

To find the pole of a given projected great circle, and the measure of its inclination to the primitive.

PROPOSITION XIV.

To measure any part of a projected great circle.

PROPOSITION XV.

To measure the projection of a spherical angle.

PROPOSITION XVI.

To project the several cases in spherical trigonometry, and measure the unknown parts of the triangle.

APPENDIX.

PROPOSITION XX.

APPENDIX.

THEOREM 1. If, from one of the points, in which a perpendicular small circle meets the primitive, a straight line be drawn to any point in the circumference of its projection, and continued to meet the diameter of the primitive that is perpendicular to the line of measures, the point of concurrence will be the pole of the projected great circle, which passes thro' the poles of the small circle, and that point in the circumference of its projection.

THEO. 2. If two equal circles, one of which is parallel, and the other inclined to the primitive, be projected, the distances of the pole of the inclined projected circle, from the centers of the projections, will be directly proportional to the radii of these projected circles.

THEOR. 3. If two equal circles, one of which is parallel, and the other inclined to the primitive, be projected, straight lines, drawn through the pole of the inclined projected circle, will intercept corresponding arches on the projection.

PROBLEM 1. Given the distance of a circle from its pole, and the projection of that pole, to describe the projection of the circle.

PROB. 2. About a given projected pole, to describe the projection of a circle that shall pass through a given point, in the plane of the primitive.

PROB. 3. To measure any part of a given projection.

PROB. 4. Through a given point, in the plane of the primitive, to describe the projection of a great circle that shall form a given angle with a given projected great circle.

PROB. 5. To describe the projection of a great circle that shall form given angles with two given projected great circles.

PROB. 6. To describe the projection of a great circle that shall have a given arch intercepted between two given projected great circles, and form a given angle with one of them.

LEMMA 1. The sine of the sum of two arches is to the sine of their difference as the sum of the tangents of these arches is to their difference.

LEM. 2. The cosine of the sum of two arches is to the cosine of their difference, as the difference between the cotangent of the one, and the tangent of the other, is to their sum.

DEMONSTRATION of PROP. VI. Book II.

Let DEF be the projection of the spherical triangle ABC, the point D being the projection of B, and the center

center of the primitive. Let EF be joined, and the tangents EG, FG, drawn. Then

$$ED + DF : ED - DF :: \text{Cot. } \frac{1}{2} EDF : T, \\ \frac{DEF - DFE}{2} \text{ or } \frac{DEG - DFG}{2}.$$

That is, $T, \frac{1}{2} AB + T, \frac{1}{2} BC : T, \frac{1}{2} AB - T, \frac{1}{2} BC$
 $:: \text{Cot. } \frac{1}{2} B : T, \frac{A - C}{2}.$

Hence $S, \frac{AB + BC}{2} : S, \frac{AB - BC}{2} :: \text{Cot. } \frac{1}{2} B :$
 $T, \frac{A - C}{2}.$

Prop. VII. may be demonstrated in like manner, by means of the second lemma.

SPHERICALS.

DEMONSTRATION of Prop. VI. Boo

Let DEF be the projection of the spherical triangle ABC, the point D being the projection of B, and

S P H E R I C S.

B O O K IV.

ORTHOGRAPHIC PROJECTION OF THE SPHERE.

IN the orthographic projection of the sphere, the projecting point is still supposed to be in the axis of a great circle, assumed as the primitive or plane of projection, but at so great a distance, that a straight line drawn from it to any point of the sphere, may be considered as perpendicular to the plane of the primitive.

The orthographic projection of any point, therefore, is where a perpendicular from that point meets the primitive.

PROPO.

PROPOSITION I

Every great circle, perpendicular to the primitive, is projected into a diameter of the primitive; and every arch of it, reckoned from the pole of the primitive, is projected into its sine.

Let BFD be the primitive, and ABCD a great circle perpendicular to it, passing through its poles A, C; then the diameter BED, which is their line of common section, will be the projection of the circle ABCD. Fig. 33.

For, if from any point as G, in the circle ABC, a perpendicular GH fall upon BD, it will also be perpendicular to the plane of the primitive. Therefore H is the projection of G. Hence the whole circle is projected into BD, and any arch AG into EH = GI, its sine.

COR. 1. Every arch of the great circle, reckoned from its intersection with the primitive, is projected into its versed sine.

COR. 2. The orthographic projection of any point on the surface of the sphere, is, within the primitive, distant from its center, by the sine of that point's distance from either pole of the primitive.

F f f

COR.

COR. 3. Every small circle, perpendicular to the primitive, is projected into its line of common section with the primitive, which is also its own diameter; and every arch of the semicircle above the primitive, reckoned from the middle point, is projected into its sine.

COR. 4. Every diameter of the primitive is the projection of a great circle, and every other chord the projection of a small circle.

COR. 5. A straight line, perpendicular to the primitive, is projected into a point, a parallel to the primitive, into an equal line, and one inclined to the primitive, into a less line, such that radius is to the cosine of the inclination, as the inclined line to its projection.

COR. 6. A spherical angle, at the pole of the primitive, also any rectilineal angle, whose plane is parallel to the primitive, is projected into an equal angle.

PROPOSITION II.

Let ELF be a great circle, inclined to the primitive FBL , and EF their line of common section. From the center C , and any other point K , in EL , a circle parallel to the primitive is projected into a circle equal to itself, and concentric with the primitive.

Let the small circle FIG be parallel to the plane of the primitive BND. The straight line HE, which joins their centers, is perpendicular to the primitive; therefore E is the projection of H. Let any radius HI and IN perpendicular to the primitive, be drawn. Then IN, HE, being parallel, are in the same plane; therefore IH, NE, the lines of common section of the plane IE, with two parallel planes, are parallel, and the figure IHEN is a parallelogram. Hence $NE = IH$, and consequently FIG is projected into an equal circle KNL, whose center is E.

COR. The radius of the projection is the cosine of the parallel's distance from the primitive, or the sine of its distance from the pole of the primitive.

PROPOSITION III.

An inclined circle is projected into an ellipse, whose transverse axis is the diameter of the circle.

CASE I. Let ELF be a great circle, inclined to the primitive EBF, and EF their line of common section. From the center C, and any other point K, in EF, let the perpendiculars CB, KI, be raised in the plane of the primitive, and CL, KN, in the plane of the great circle, meeting the circumference in L, N. Let LG, ND, be perpendicular to CB, KI; then G, D are

are the projections of L, N . And, because the tri-angles LOG, NKD are equiangular, $CL = CG$, $NK = DK$, or $EG = CG = EK = DK$; there-fore the points G, D , are in the curve of an ellipse, of which EF is the transverse, and CG the semi-con-
gate axis. ON, TO or LM is perpendicular to OT, NO

Fig. 36.

Cor. 1. In a projected great circle, the semi-con-
gate axis is the cosine of the great circle's inclination to the primitive.

Cor. 2. Perpendiculars to the transverse axis inter-
cept corresponding arches of the projection and the primitive.

Cor. 3. The eccentricity of the projection is the
sine of the great circle's inclination to the primitive.

Fig. 37.

CASE 2. Let AQB be a small circle, inclined to
the primitive, and let the great circle LBM , perpen-
dicular to both, intersect them in the lines AB, LM .
From the center O , and any other point N in the
diameter AB , let the perpendiculars POP, NQ be
drawn in the plane of the small circle, to meet its
circumference in P, Q . Also, from the points
 A, N, O, B , let AG, NP, OC, BH , be drawn per-
pendicular to LM , and from P, Q, T, PE, QD, TF ,
perpendicular to the primitive; then G, I, C, H, E ,
 D, F , are the projections of these points. Because
 OP is perpendicular to EBM , and OC, PE , being
perpendicular to the primitive, are in the same plane,

$P R O P O$

the

the plane COPE is perpendicular to LBM. But the primitive is perpendicular to LBM. Therefore the line of common section EC is perpendicular to LBM, and to LM. Hence CP is a parallelogram, and $EC = OP$. In like manner, EC, DI , are proved perpendicular to LM, and equal to OT, NQ. Thus, ECF is a straight line, and equal to the diameter PT.

Let QR, DK, be parallel to AB, LM; then $RO = NQ = DI = KC$, and $PR \times RT = EK \times KF$. But $AO : CG :: NO : CI$; therefore $AO : CG :: QR : DK$, and $EC : CG :: EK : DK$.

COR. 1. The transverse axis is to the conjugate, as radius to the cosine of the circle's inclination to the primitive.

COR. 2. Half the transverse axis is the cosine of half the sum of the greatest and least distances of the small circle from the primitive.

COR. 3. The extremities of the conjugate axis are in the line of measures distant from the center of the primitive, by the cosines of the greatest and least distances of the small circle from the primitive.

COR. 4. If, from the extremities of the conjugate axis, of any elliptical projection, perpendiculars be raised, (in the same direction, if the circle do not intersect the primitive, but, if otherwise, in opposite directions,) they will intercept an arch of the primitive, whose chord is equal to the circle's diameter.

perpendicular to the primitive, are in the same plane.
the
P R O P O.

PROPOSITION IV.

The projected poles of an inclined circle are, in its line of measures, distant from the center of the primitive, the sine of the circle's inclination to the primitive.

Fig. 38. Let ABCD be a great circle, perpendicular, both to the primitive and the inclined circle, and intersecting them in the diameters AC, MN. Then ABCD passes through the poles of the inclined circle; let these be P, Q, and let Pp, Qq, be perpendicular to AC, p, q are the projected poles, and it is evident that $pO = S, BP$, or MA , the inclination.

COR. II. The center of the primitive, the center of projection, the projected poles, and the extremes of the conjugate axis, are all in one and the same straight line.

COR. II. As radius is to the sine of a small circle's inclination to the primitive, so is the cosine of its distance from its own pole to the distance of the center of its projection from the center of the primitive.

A P P E N D I X.

P R O P O .

THEOREM I. In any inclined circle two diameters, which cut each other at right angles, are projected into conjugate diameters of the ellipse.

THEOREM

PROPOSITION V.

To describe the projection of a small circle parallel to the primitive, its distance from the pole of the primitive being given.

PROPOSITION VI.

To project an inclined circle, whose distance from its pole and inclination to the primitive, are given.

PROPOSITION VII.

To find the poles of a given projection.

PROPOSITION VIII.

To measure any part of a given projection.

A P P E N D I X.

THEOREM I.

In any inclined circle, two diameters, which cut each other at right angles, are projected into conjugate diameters of the ellipse.

THEOR.

THEOR. 2. In every elliptical projection, half the transverse axis is to the excentricity, as radius to the sine of the circle's inclination to the primitive; and half the conjugate axis is to the excentricity, as radius to the tangent of the same.

PROBLEM 1. Through two given points, in the plane of the primitive, to describe the projection of a great circle.

PROB. 2. Given the distance of a circle from its pole, and the projection of that pole, to describe the projection of the circle.

PROB. 3. Through a given point, in the plane of the primitive, to describe the projection of a great circle, having a given inclination to the primitive.

DEMONSTRATION of the last THEOR.

Book II.

Let the spherical triangle ABC be projected orthographically upon ABD, the plane of one of its sides; and let the projections of the other two sides be continued to meet the primitive in F, D. Let AEF, BED, the transverse axes of the ellipses ACF, BCD, be drawn, also AK, LCM, GEI, perpendicular to BD,

BD, and LP, CO perpendicular to AF. Then AK, LM are the sines of the two sides, GH the versed sine of the included angle, and PO the excess of the versed sine of the base, above the versed sine of AL, the difference of the two sides ;

But $GE : LM :: GH : LC.$

And $AE : AK :: LC : PO.$

Therefore $AE \times GE : AK \times LM :: GH : PO.$

That is $R^2 : S, AB \times S, BC :: V, B : V, AC - V,$
 $\frac{AB - BC.}{}$

F I N I S.

BD, and IC, are the lines of the circle. LM are the lines of the included angle, and PO the vertices of the circle. Above the vertex of AL, the distance of the two lines;

Bar GE: LM: GH: IC.
 And AE: AL: IC: PO.
 Therefore AE x GE: AL x LM: IC x PO.
 That is R: S: AB: V: B: H: AC: V:

AB	IX
K	X
XI	XI
XII	XII
XIII	XIII

CONIC SECTIONS

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MEASUREMENT

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SPHERICS

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FIG. 31.

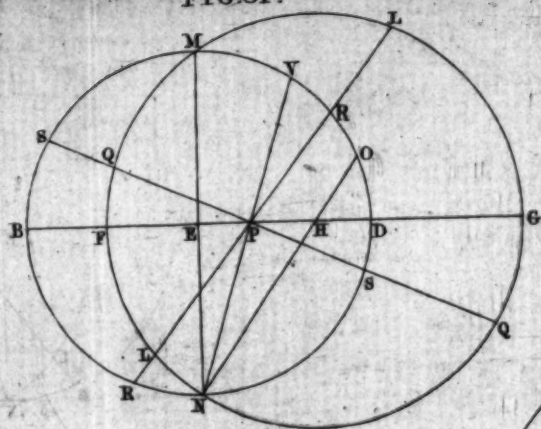


FIG. 32.

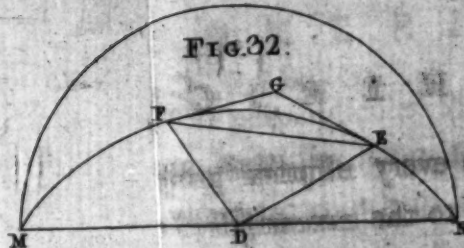


FIG. 33.

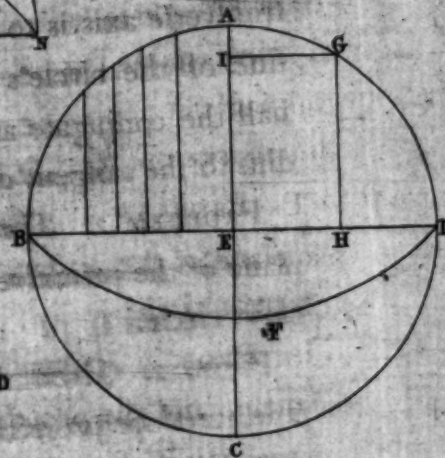


FIG. 35.

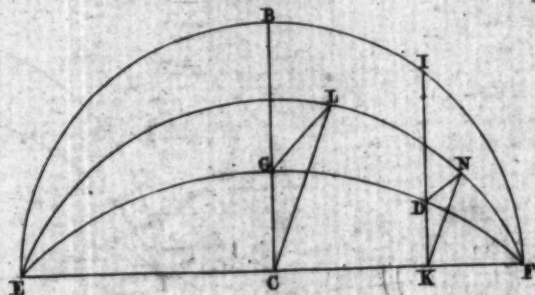


FIG. 34.

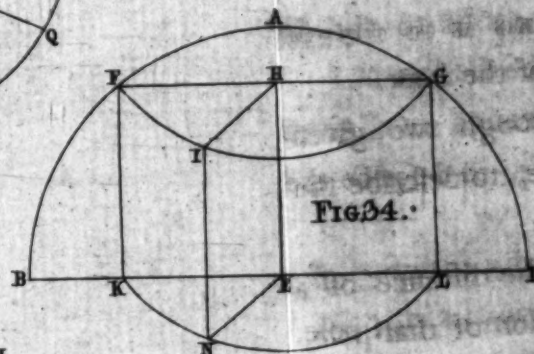


FIG. 37.

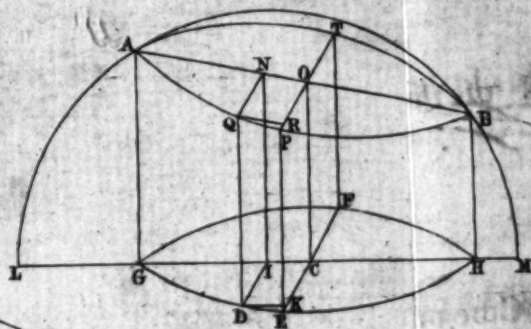


FIG. 36.

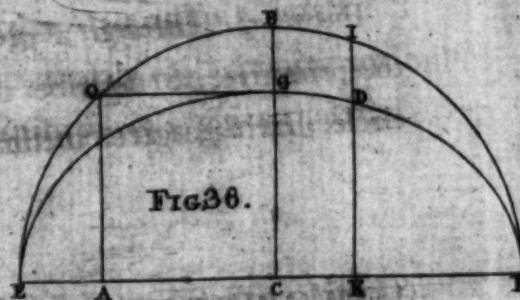


FIG. 38.

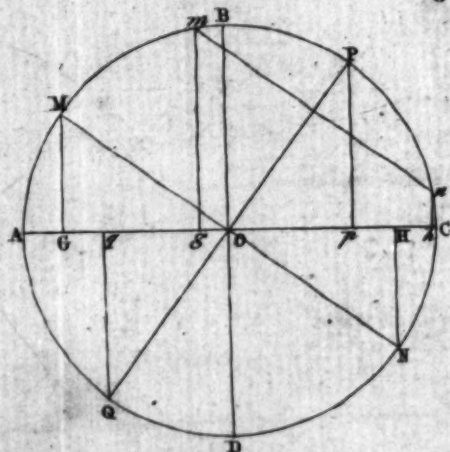
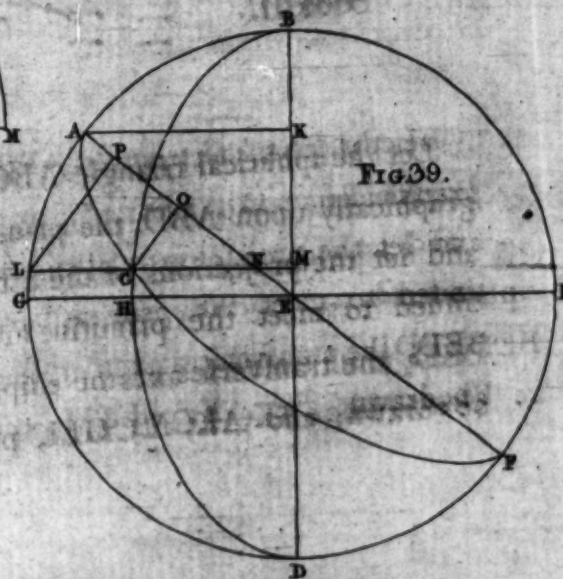


FIG. 39.



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E R R A T A.

Page 2. In def. 12. *for a surface, read a plane surface.*

- 7. line 7. *for equal read equal °.*
- 8. dele °,
- 10. *for ax. 4. read ax. 3.*
- 8. 11. *for DEF read EDF.*
- 19. *for ax. 2. read ax. 1.*
- 9. 2. *for ax. 2. read ax. 1.*
- 29. ult. *for ABD read ADB.*
- 40. 2. *for 10. 1. read 9. 1.*
- 41. 4. *for square read squares,*
- 46. 9. *for BCA. read CBA.*
- 59. 16. *for cor. 24. read cor. 23.*
- 61. 4. *for 12. 1. read 11. 1.*
- 62. 23. *for 11. 1. read 10. 1.*